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Markov Property of Brownian Motion

Exercise 6.1.4. *Show that the standard* 1*-dimensional Brownian motion has the time homogeneous Markov property, see for example* [1, p. 177–178] *and* [9, Thm. 3.9]*. Note that the same result holds for the standard N-dimensional Brownian motion as well.*

The N-dimensional Brownian motion defined by $\mathcal{B} := (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (B_t)_{t\geq 0})$ taking value in $\mathbb{R}^N$. Consider a bounded and measurable function $f: \mathbb{R}^N \to \mathbb{R}$ that is $\mathcal{F}_s$-measurable for $s<t$. Let us define $\psi : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ with $\psi(B_s, B_t - B_s) = f(B_s + B_t - B_s) = f(B_t)$ be a measurable function.

Since $B_t - B_s$ is independent of $\mathcal{F}_s$ by definition with $\mathbb{E}(B_s) = 0$ and $\mathbb{E}(B_t - B_s) = 0$ (belonging to $L^1(\Omega, \mathcal{F}_s, \mathbb{P})$ and $L^1(\Omega, \sigma(B_t - B_s), \mathbb{P})$ respectively) and $\psi(B_s, B_t - B_s) = f(B_t)$ is absolutely integrable (f is bounded and measurable), then, by using the freezing lemma (lemma 3.1.6), we obtain

$$
\begin{aligned}
\mathbb{E}(f(B_t) \mid \mathcal{F}_s) &= \mathbb{E}(f(B_s + B_t - B_s) \mid \mathcal{F}_s) \\
&= \mathbb{E}(\psi(B_s, B_t - B_s) \mid \mathcal{F}_s) \\
&= \mathbb{E}(\psi(B_s, B_t - B_s) \mid B_s) \\
&= \mathbb{E}(f(B_t) \mid B_s).
\end{aligned}
$$

freezing lemma

Hence, the N-dimensional Brownian motion has the Markov property.
For the Brownian motion with $s<t$, the process $\{B_t \mid t>s\}$ given $B_s = x \in \mathbb{R}^N$ a.s. is equivalent to the process $\{B_{t-s} \mid t>s\}$ given $B_0 = x$ a.s. Then, by the definition of the Markov property, we can also express the equation for any $A \in \sigma(\mathbb{R}^N)$ as

$$\mathbb{P}(B_t \in A \mid \mathcal{F}_s) = \mathbb{P}(B_t \in A \mid B_s) = \mathbb{P}(B_{t-s} \in A \mid B_0) \quad \text{a.s.}$$

Therefore, the N-dimensional Brownian motion has the time homogeneous Markov property. $\square$