Exercise 1.1.6

HADIKO Rifqi Aufa Sholih
062101868

Exercise 1.1.6. *If* $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space and if* $A, B \in \mathcal{F}$, *check that*

1) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$, *where* $A^c := \Omega \setminus A$,

2) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$,

3) If $A \subset B$, *then* $\mathbb{P}(A) \leq \mathbb{P}(B)$.

Consider the definition of a probability space

Definition 1.1.5 (Probability space)**.** *A probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *consists of a measurable space* $(\Omega, \mathcal{F})$ *and a function* $\mathbb{P} : \mathcal{F} \to [0, 1]$ *satisfying* $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\emptyset) = 0$ *and*

$$\mathbb{P}\Big(\bigcup_{j \in \mathbb{N}} A_j\Big) = \sum_{j \in \mathbb{N}} \mathbb{P}(A_j)$$

whenever $A_j \cap A_k = \emptyset \ \forall j \neq k$. *We call* Ω the sample space, $\mathcal{F}$ the event space, $\omega \in \Omega$ an elementary event *and* $A \in \mathcal{F}$ an event, *and finally* $\mathbb{P}$ the probability measure.

Proof:

1.) Consider a set $A^c := \Omega \setminus A$. Since $A \cap A^c = \emptyset$ and $A \cup A^c = \Omega$, then by using the definition of a probability space, we get

$$\mathbb{P}(\Omega) = \mathbb{P}(A \cup A^c)$$
$$\iff \quad 1 = \mathbb{P}(A) + \mathbb{P}(A^c)$$
$$\iff \quad \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$

Hence, $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ with $A^c := \Omega \setminus A$.

2.) For $A, B \in \mathcal{F}$, consider the following relations:

i.) $A \cup B = (A \setminus (A \cap B)) \cup B$

ii.) $(A \setminus (A \cap B)) \cap B = \emptyset$

iii.) $A = (A \setminus (A \cap B)) \cup (A \cap B)$

iv.) $(A \cap B) \cap (A \setminus (A \cap B)) = \emptyset$

By using definition 1.1.5, relation (i.) and (ii.), we obtain

$$\mathbb{P}(A \cup B) = \mathbb{P}((A \setminus (A \cap B)) \cup B) = \mathbb{P}(A \setminus (A \cap B)) + \mathbb{P}(B)$$

Then, by using definition 1.1.5, relation (iii.) and (iv), we get

$$\mathbb{P}(A) = \mathbb{P}((A \setminus (A \cap B)) \cup (A \cap B)) = \mathbb{P}(A \setminus (A \cap B)) + \mathbb{P}(A \cap B)$$

$$\iff \quad \mathbb{P}(A \setminus (A \cap B)) = \mathbb{P}(A) - \mathbb{P}(A \cap B) .$$

Then, by using the results above, we obtain

$$\begin{aligned} \mathbb{P}(A \cup B) &= \mathbb{P}((A \setminus (A \cap B)) \cup B) \\ &= \mathbb{P}(A \setminus (A \cap B)) + \mathbb{P}(B) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) . \end{aligned}$$

Therefore, we obtained

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) .$$

3.) If $A \subset B$, since $B = A \cup (B \setminus A)$ and $A \cap (B \setminus A) = \phi$, with $B \setminus A$ well defined for $A \subset B$, then by using the fact that $\mathbb{P}(B \setminus A) \in [0,1]$ by definition, we obtain

$$\mathbb{P}(B) = \mathbb{P}(A \cup (B \setminus A)) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$$

$$\iff \quad \mathbb{P}(B \setminus A) = \mathbb{P}(B) - \mathbb{P}(A) \geq 0$$

$$\iff \quad \mathbb{P}(B) \geq \mathbb{P}(A) .$$

Hence, if $A \subset B$, $\mathbb{P}(A) \leq \mathbb{P}(B)$. $\square$