Several Exercises on Itô Integral Section

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Exercise 4.2.13.

Study the equality

$$\int_a^b B_t \, \mathrm{d}B_t = \frac{1}{2}\big(B_b^2 - B_a^2 - (b-a)\big).$$

The exercise above is based from ([2], pg. 36).

The results below are mainly based on ([1], **Example 4.2**). For $n \in \mathbb{N}$, consider a partition $\mathcal{P}_n$ defined by $a = t_{n,0} < t_{n,1} < t_{n,2} < \cdots < t_{n,n} = b$, and let

$$X_{n,t} = \sum_{i=0}^{n-1} B_{t_{n,i}} \mathbf{1}_{(t_{n,i}, t_{n,i+1}]}(t)$$

with $(X_{n,t})_{t \in [a,b]}$ is an adapted elementary process for all n and the equation above is just the same equation as Equation (4.2.2) ([2], pg. 34). Note that $B_{t_{n,i}}$ is $\mathcal{F}_{t_{n,i}}$-measurable because Brownian process is a stochastic process. By the continuity of B_t, one has $\lim_{n\to\infty} X_{n,t} = B_t$ a.s. since one has $\sup_i (t_{n,i+1} - t_{n,i}) \to 0$. Thus, according to **Definition 4.2.6** ([2], pg. 34) the Itô Integral of $X_{n,t}$ from a to b is given by

$$\int_a^b X_{n,t} \, \mathrm{d}B_t = \sum_{i=0}^{n-1} B_{t_{n,i}} \big(B_{t_{n,i+1}} - B_{t_{n,i}}\big).$$

Observe that one has the following equality

$$B_{t_{n,i}} \big(B_{t_{n,i+1}} - B_{t_{n,i}}\big) = \frac{1}{2}\Big(B_{t_{n,i+1}}^2 - B_{t_{n,i}}^2 - \big(B_{t_{n,i+1}} - B_{t_{n,i}}\big)^2\Big).$$

Hence, the Itô Integral becomes in the following form

$$\int_a^b X_{n,t} \, \mathrm{d}B_t = \frac{1}{2} \sum_{i=0}^{n-1} \Big(B_{t_{n,i+1}}^2 - B_{t_{n,i}}^2\Big) - \frac{1}{2} \sum_{i=0}^{n-1} \big(B_{t_{n,i+1}} - B_{t_{n,i}}\big)^2.$$

The first summation on the right-hand side can be easily computed as follows

$$\sum_{i=0}^{n-1} \Big(B_{t_{n,i+1}}^2 - B_{t_{n,i}}^2\Big) = B_{t_{n,n}}^2 - B_{t_{n,n-1}}^2 + B_{t_{n,n-1}}^2 - \cdots + B_{t_{n,1}}^2 - B_{t_{n,0}}^2 = B_b^2 - B_a^2.$$

The summation computed above is telescopic. To compute the second summation on the right-hand side, one cannot use the same trick used in computing the first summation. Let us recall **Theorem 2.4.8.** ([2], pg. 19).

Theorem: Properties of 1-dimensional Brownian motion

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, (B_t)_{t\in\mathbb{R}^+})$ be a 1-dimensional Brownian motion.

1. Almost every path has an infinite variation on any finite interval, namely for any $a, b \in \mathbb{R}^+$,
$$\mathbb{P}\left(\left\{\omega \in \Omega \,\middle|\, \mathrm{var}_{[a,b]}(t \mapsto B_t(\omega)) = \infty\right\}\right) = 1.$$
2. The quadratic variation of the Brownian motion converges in the L^2-sense, namely
$$\lim_{|\mathcal{P}_\ell|\to 0} \mathbb{E}\left(\left[\sum_{j=0}^{n_\ell-1} (B_{t_{\ell,j+1}} - B_{t_{\ell,j}})^2 - (b-a)\right]^2\right) = 0.$$
where $|\mathcal{P}_\ell| := \max_{j\in\{0,1,2,\ldots,n_\ell-1\}} |t_{\ell,j+1} - t_{\ell,j}|$.
3. Almost every path is nowhere differentiable, namely
$$\mathbb{P}\left(\left\{\omega \in \Omega \,\middle|\, t \to B_t(\omega) \text{ is nowhere differentiable}\right\}\right) = 1.$$

Consider the second point of the theorem above. If one takes the limit $|\mathcal{P}_\ell| \to 0$, then it corresponds to taking the limit $n_\ell \to \infty$. Then, by using the second point of the theorem above (The quadratic variation of the Brownian motion converges in the L^2-sense), one has
$$\int_a^b B_t \, \mathrm{d}B_t = \lim_{n\to\infty} \int_a^b X_{n,t} \, \mathrm{d}B_t = \frac{1}{2}\big(B_b^2 - B_a^2 - (b-a)\big).$$

Theorem 4.2.14. ([2], pg. 36)

Let $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (B_t)_{t\geq 0}\right)$ *be the standard* 1*-dim. Brownian motion, and let* $(X_t)_{t\in[0,T]}$, $(Y_t)_{t\in[0,T]}$ *be adapted stochastic processes belonging to* $M^2([0,T])$*. Then*
$$\mathbb{E}\left(\left(\int_0^T X_t \, \mathrm{d}B_t\right)\left(\int_0^T Y_t \, \mathrm{d}B_t\right)\right) = \int_0^T \mathbb{E}(X_t Y_t) \, \mathrm{d}t.$$

Let $I_1 := \int_0^T X_t \, \mathrm{d}B_t$ and $I_2 := \int_0^T Y_t \, \mathrm{d}B_t$. Observe that $I_1 I_2$ can be expressed as follows
$$I_1 I_2 = \frac{(I_1+I_2)^2}{2} - \frac{I_1^2}{2} - \frac{I_2^2}{2} = \frac{1}{2}\left(\int_0^T X_t \, \mathrm{d}B_t + \int_0^T Y_t \, \mathrm{d}B_t\right)^2 - \frac{1}{2}\left(\int_0^T X_t \, \mathrm{d}B_t\right)^2 - \frac{1}{2}\left(\int_0^T Y_t \, \mathrm{d}B_t\right)^2.$$

Then, by using the isometry property of Itô Integral written in **Proposition 4.2.7** ([2], pg. 35), namely
$$\mathbb{E}\left(\left(\int_a^b X_t \, \mathrm{d}B_t\right)^2 \middle| \mathcal{F}_a\right) = \mathbb{E}\left(\int_a^b X_t^2 \, \mathrm{d}B_t \middle| \mathcal{F}_a\right).$$

By using the equation written above for $a = 0$, $b = T$ and by the linearity of the expectation, one has

$$
\begin{aligned}
&\mathbb{E}\biggl(\Bigl(\int_0^T X_t \,\mathrm{d}B_t\Bigr)\Bigl(\int_0^T Y_t \,\mathrm{d}B_t\Bigr)\biggr) \\
&= \frac{1}{2}\mathbb{E}\biggl(\Bigl(\int_0^T X_t \,\mathrm{d}B_t + \int_0^T Y_t \,\mathrm{d}B_t\Bigr)^2\biggr) - \frac{1}{2}\mathbb{E}\biggl(\Bigl(\int_0^T X_t \,\mathrm{d}B_t\Bigr)^2\biggr) - \frac{1}{2}\mathbb{E}\biggl(\Bigl(\int_0^T Y_t \,\mathrm{d}B_t\Bigr)^2\biggr) \\
&= \frac{1}{2}\mathbb{E}\biggl(\Bigl(\int_0^T (X_t + Y_t) \,\mathrm{d}B_t\Bigr)^2\biggr) - \frac{1}{2}\mathbb{E}\biggl(\Bigl(\int_0^T X_t \,\mathrm{d}B_t\Bigr)^2\biggr) - \frac{1}{2}\mathbb{E}\biggl(\Bigl(\int_0^T Y_t \,\mathrm{d}B_t\Bigr)^2\biggr) \\
&= \frac{1}{2}\int_0^T \mathbb{E}\bigl((X_t + Y_t)^2\bigr) \,\mathrm{d}t - \frac{1}{2}\int_0^T \mathbb{E}\bigl(X_t^2\bigr) \,\mathrm{d}t - \frac{1}{2}\int_0^T \mathbb{E}\bigl(Y_t^2\bigr) \,\mathrm{d}t \\
&= \frac{1}{2}\int_0^T \mathbb{E}\bigl(X_t^2 + Y_t^2 + 2X_tY_t\bigr) \,\mathrm{d}t - \frac{1}{2}\int_0^T \mathbb{E}\bigl(X_t^2\bigr) \,\mathrm{d}t - \frac{1}{2}\int_0^T \mathbb{E}\bigl(Y_t^2\bigr) \,\mathrm{d}t \\
&= \frac{1}{2}\int_0^T \bigl(\mathbb{E}\bigl(X_t^2\bigr) + \mathbb{E}\bigl(Y_t^2\bigr) + 2\mathbb{E}(X_tY_t)\bigr) \,\mathrm{d}t - \frac{1}{2}\int_0^T \mathbb{E}\bigl(X_t^2\bigr) \,\mathrm{d}t - \frac{1}{2}\int_0^T \mathbb{E}\bigl(Y_t^2\bigr) \,\mathrm{d}t \\
&= \int_0^T \mathbb{E}(X_tY_t) \,\mathrm{d}t.
\end{aligned}
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References

[1] Fima C Klebaner. *Introduction to stochastic calculus with applications*. World Scientific Publishing Company, 2012.

[2] Serge Richard. *Special Mathematics Lecture: Introduction to Stochastic Calculus*. 2023.