1-Dimensional Brownian Processes are Gaussian Processes

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Exercise 2.4.3.

Show that the Brownian process is a Gaussian process.

Proof:

Let us first write the definition of 1-dimensional Brownian motion (Definition 2.4.1) from the lecture note

> Definition 2.4.1 (1-dimensional Brownian motion)
>
> A Stochastic process $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ taking values in $\mathbb{R}$ is a 1-dimensional Brownian motion if
>
> (i) $B_0 = \mathbf{0}$ a.s. (almost surely)
>
> (ii) For any $0 \le s \le t$ the random variable $B_t - B_s$ is independent of $\mathcal{F}_s$
>
> (iii) For any $0 \le s \le t$ the random variable $B_t - B_s$ is a Gaussian random variable $N(0, t-s)$.

From the definition above, we know that $B_t - B_s$ is a Gaussian r.v. for any $0 \le s \le t$ as the random variable behaves in accord with the normal distribution $N(0, t-s)$. To show that a Brownian process is a Gaussian process, we need to show that $B_{t_1}, B_{t_2}, \dots, B_{t_m}$ are jointly Gaussian, namely we must show that $\alpha_1 B_{t_1} + \cdots + \alpha_m B_{t_m}$ is a Gaussian random variable for arbitrary $\alpha_1, \dots, \alpha_m \in \mathbb{R}$ and for $0 \le t_1 < t_2 < \cdots < t_m$.

First, we know that B_{t_1} is a Gaussian random variable because from (i) & (iii) of Definition 2.4.1, we know that $B_{t_1} - B_0 = B_{t_1}$ is a Gaussian random variable and is independent of $\mathcal{F}_0$ by (ii). The strategy to prove the proposition above is by using proof by induction. Let us consider the case $m = 2$. For $m = 2$, we have the following expression

$$\alpha_1 B_{t_1} + \alpha_2 B_{t_2} = (\alpha_1 + \alpha_2) B_{t_1} + \alpha_2 (B_{t_2} - B_{t_1})$$

Now, we know that $(\alpha_1 + \alpha_2) B_{t_1}$ is a Gaussian random variable measurable with respect to $\mathcal{F}_{t_1}$ and we also know that $\alpha_2(B_{t_2} - B_{t_1})$ is a Gaussian random variable independent of $\mathcal{F}_{t_1}$ by (ii). Hence, we know that $\alpha_1 B_{t_1} + \alpha_2 B_{t_2}$ is a Gaussian random variable. Let us assume that it is true for $m - 1$

and let us prove the case m

$$\alpha_1 B_{t_1} + \alpha_2 B_{t_2} + \cdots + \alpha_m B_{t_m} = (\alpha_1 B_{t_1} + \alpha_2 B_{t_2} + \cdots + (\alpha_{m-1} + \alpha_m) B_{t_{m-1}}) + \alpha_m (B_{t_m} - B_{t_{m-1}})$$

We know that $(\alpha_1 B_{t_1} + \alpha_2 B_{t_2} + \cdots + (\alpha_{m-1} + \alpha_m) B_{t_{m-1}})$ is a Gaussian random variable by the induction assumption and is measurable with respect to $\mathcal{F}_{t_{m-1}}$. Then, we also know that $\alpha_m (B_{t_m} - B_{t_{m-1}})$ is a Gaussian random variable by (iii) of Definition 2.4.1 and is independent of $\mathcal{F}_{t_{m-1}}$ by (ii) of Definition 2.4.1. Hence, $B_{t_1}, B_{t_2}, \ldots, B_{t_m}$ are jointly Gaussian and therefore the Brownian process is a Gaussian process.

$\square$