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Summary of the Ornstein-Uhlenbeck process

A. Stationery property

The Ornstein-Uhlenbeck process Y_t is the Gaussian process with mean $\mathbb{E}[Y_t] = 0$ and if it starts with $Y_0 = 0$, the covariance is of the form:

$$\mathrm{Cov}(Y_s, Y_t) = \frac{1}{2} e^{-(t-s)}(1 - e^{-2s}) \quad , \text{ for } s \leq t$$

However, if the starting point Y_0 is Gaussian with mean 0 and variance $\frac{1}{2}$, we have $\mathbb{E}[Y_t] = 0$ and the covariance is of the form:

$$\mathrm{Cov}(Y_s, Y_t) = \frac{1}{2} e^{-(t-s)} \quad , \text{ for } s \leq t$$

We can see that the covariance only depends on the difference of time.

This means that if we shift this process Y_t to Y_{t+a}, where $a \geq 0$, the process will still have the same distribution. This is called a stationery property.

B. As an Ito integral

We can write the Ornstein-Uhlenbeck process as,

$$Y_t = e^{-t} \int_0^t e^s \, dB_s \ , \ t \geq 0$$

with mean 0 and the covariance,

$$\mathbb{E}[Y_t Y_s] = e^{-t-s} \int_0^s e^{2u} \, du = \frac{1}{2}(e^{-(t-s)} - e^{-(t+s)}) = \frac{1}{2} e^{-(t-s)}(1 - e^{-2s}) \ , \ s \leq t.$$

We can also start the process at Y_0, a Gaussian random variable of mean 0 and variance $\frac{1}{2}$ independent of the Brownian motion B_t. The process then takes the form,

$$Y_t = Y_0 e^{-t} + e^{-t} \int_0^t e^s \, dB_s$$

Since Y_0 and the Ito integral are independent by assumption, then

$$\mathbb{E}[Y_t Y_s] = \frac{1}{2} e^{-t-s} + \frac{1}{2}(e^{-(t-s)} - e^{-(t+s)}) = \frac{1}{2} e^{-(t-s)} \ , \ s \leq t$$

C. As a stochastic differential equation

For the Ito integral of the Ornstein-Uhlenbeck process in the previous section,

$$Y_t = Y_0 e^{-t} + e^{-t} \int_0^t e^s dB_s$$

note that this process is an explicit function of t, and of the Ito process $X_t = Y_0 + \int_0^t e^s dB_s$. So, we have $Y_t = f(t, X_t)$ with $f(t,x) = e^{-t}x$, and this is not in an explicit function of t and B_t so the Ito's formula is not directly applicable.

Since $\partial_1 f = e^{-t}$, $\partial_1^2 f = 0$, and $\partial_0 f = -f$, we get from Ito's formula for Ito processes,

$$df(t, X_t) = \partial_1 f(t, X_t)\, dX_t + \left(\partial_0 f(t, X_t) + \frac{1}{2}\partial_1^2 f(t, X_t)\, e^{2t}\right) dt$$

$$dY_t = e^{-t} dX_t + \left(-f(t, X_t) + 0\right) dt$$

$$dY_t = e^{-t} dX_t - Y_t\, dt = dB_t - Y_t\, dt$$

This differential equation is in fact has a nice interpretation: the drift is positive if $Y_t < 0$ and negative if $Y_t > 0$, and proportional to the position. This is the mechanism that ensures the process to not venture too far from 0 and is eventually stationery.

We can also add two parameters for the volatility and the drift,

$$dY_t = -k Y_t\, dt + \sigma\, dB_t \,, \quad k \in \mathbb{R}, \sigma > 0$$

and the solution for this stochastic differential equation is,

$$Y_t = Y_0 e^{-kt} + e^{-kt} \int_0^t e^{ks} \sigma\, dB_s$$

Note that if $k < 0$, the solution doesn't converge to a stationery distribution.

D. References

A First Course in Stochastic Calculus. Louis-Pierre Arguin. p 38-39, 113, 159.