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Exercise 1.1.13 (Classical probability distributions). *Recall the definition of a few classical probability distributions, and recast them in the framework and with the terminology introduced above. For example, consider the Bernoulli distribution, the binomial distribution, the Poisson distribution, the uniform distribution, the exponential distribution, etc.*

BERNOULLI DISTRIBUTION (parameter $p \in [0,1]$)

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable space $(\Lambda, \mathcal{E}) = (\{0,1\}, P(\{0,1\}))$, where $P(\{0,1\})$ is the power set of $\{0,1\}$.

A random variable X with a Bernoulli distribution is a measurable function from $(\Omega, \mathcal{F})$ to $(\Lambda, \mathcal{E})$, which means that for any $A \in P(\{0,1\})$, $\{\omega \in \Omega \mid X(\omega) \in A\} \equiv X^{-1}(A) \in \mathcal{F}$. Observe also that $X(\Omega) = P(\{0,1\})$.

The corresponding induced probability measure is the map $\mu_X : P(\{0,1\}) \to [0,1]$ defined by for any $A \in P(\{0,1\})$

$$\mu_X(A) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in A\}) = \mathbb{P}(X^{-1}(A)) \equiv \mathbb{P}(X \in A)$$

Also, the corresponding probability mass function is the function $p_X : P(\{0,1\}) \to \{0, p, 1-p, 1\}$ by $p_X(x) = \mathbb{P}(X^{-1}(\{x\}))$, which in detail, $p_X(\{\phi\}) = 0$, $p_X(\{1\}) = p$, $p_X(\{0\}) = 1-p$, and $p_X(\{0,1\}) = 1$ for a particular $p \in [0,1]$.

BINOMIAL DISTRIBUTION (parameter $n \in \mathbb{Z}_+$, $p \in [0,1]$)

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable space $(\Lambda, \mathcal{E})$, where $\Lambda = \{0, 1, \dots, n\}$, and $\mathcal{E} = P(\Lambda)$.

A random variable X with a binomial distribution is a measurable function from $(\Omega, \mathcal{F})$ to $(\Lambda, \mathcal{E})$, which means that for any $A \subset \{0, \dots, n\}$, $\{\omega \in \Omega \mid X(\omega) \in A\} \equiv X^{-1}(A) \in \mathcal{F}$. Also, $X(\Omega) = \{0, \dots, n\}$.

The corresponding induced probability measure is the map $\mu_X : \{0, 1, \dots, n\} \to [0,1]$ defined by for any $A \subset \{0, 1, \dots, n\}$,

$$\mu_X(A) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in A\}) = \mathbb{P}(X^{-1}(A)) \equiv \mathbb{P}(X \in A)$$

Also, the corresponding probability mass function is the function $p_X : P(\{0, 1, \dots, n\}) \to [0,1]$ defined by $p_X(x) = \mathbb{P}(X^{-1}(\{x\}))$, which satisfy $p_X(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$ for a particular $p \in [0,1]$.

POISSON DISTRIBUTION (parameter $\lambda \in \mathbb{R}_+$)

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable space $(\Lambda, \mathcal{E})$, where $\Lambda = \mathbb{Z}_+$ and $\mathcal{E} = P(\Lambda)$.

A random variable X with a Poisson distribution is a measurable function from $(\Omega, \mathcal{F})$ to $(\Lambda, \mathcal{E})$, which means that for any $A \in \mathbb{Z}_+$ $\{\omega \in \Omega \mid X(\omega) \in A\} \equiv X^{-1}(A) \in \mathcal{F}$. Also, $X(\Omega) = \mathbb{Z}_+$.

The corresponding induced probability measure is the map $\mu_X : \mathbb{Z}_+ \to [0,1]$ defined by for any $A \in P(\Lambda)$,

$$\mu_X(A) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in A\}) = \mathbb{P}(X^{-1}(A)) \equiv \mathbb{P}(X \in A)$$

Also, the corresponding probability mass function is the function $p_X : P(\Lambda) \to [0,1]$ defined by $p_X(x) = \mathbb{P}(X^{-1}(\{x\}))$, which satisfy $p_X(x) = \frac{1}{x!} \lambda^x e^{-\lambda}$ for a particular $\lambda \in \mathbb{R}_+$.

UNIFORM DISTRIBUTION (parameter $a < b$ in $\mathbb{R}$)

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable space $(\Lambda, \mathcal{E})$, where $\Lambda = \mathbb{R}$, $\mathcal{E} = \sigma_B$.

A random variable X with a uniform distribution is a measurable function from $(\Omega, \mathcal{F})$ to $(\Lambda, \mathcal{E})$, which means that for any $A \in \sigma_B$ $\{\omega \in \Omega \mid X(\omega) \in A\} \equiv X^{-1}(A) \in \mathcal{F}$.

The corresponding induced probability measure is the map $\mu_X : \mathbb{R} \to [0,1]$ defined by for any $A \in \sigma_B$,

$$\mu_X(A) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in A\}) = \mathbb{P}(X^{-1}(A)) \equiv \mathbb{P}(X \in A)$$

Also, the corresponding probability density function is the function $\pi_X : \mathbb{R} \to [0,\infty)$ that satisfies $\pi_X(x) = \begin{cases} \frac{1}{b-a}, & \text{for } a \le x \le b \\ 0, & \text{for } x < a \text{ or } x > b. \end{cases}$ for a particular $a, b \in \mathbb{R}$ with $b > a$, which correlates with the induced probability measure as $\mu_X(A) = \int_A \pi_X(x)\,dx$, $A \in \sigma_B$.

EXPONENTIAL DISTRIBUTION (parameter $\lambda \in \mathbb{R}_+$)

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a measurable space $(\Lambda, \mathcal{E})$, where $\Lambda = \mathbb{R}$, $\mathcal{E} = \sigma_B$.

A random variable X with an exponential distribution is a measurable function from $(\Omega, \mathcal{F})$ to $(\Lambda, \mathcal{E})$, which means that for any $A \in \sigma_B$ $\{\omega \in \Omega \mid X(\omega) \in A\} \equiv X^{-1}(A) \in \mathcal{F}$.

The corresponding induced probability measure is the map $\mu_X : \mathbb{R} \to [0,1]$ defined by for any $A \in \sigma_B$,

$$\mu_X(A) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) \in A\}) = \mathbb{P}(X^{-1}(A)) \equiv \mathbb{P}(X \in A)$$

Also, the corresponding probability density function is the function $\pi_X : \mathbb{R} \to [0,\infty)$ that satisfies $\pi_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x \ge 0. \\ 0, & \text{for } x < 0 \end{cases}$ for a particular $\lambda \in \mathbb{R}_+$, which correlates with the induced probability measure as $\mu_X(A) = \int_A \pi_X(x)\,dx$, $A \in \sigma_B$.

REFERENCE

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