Gambler's Ruin problem with Brownian Motion without Drift

Tran Quynh Le Phuong

January 24, 2024

1 Introduction

This report studies the Gambler's Ruin problem with Brownian Motion without Drift with the following description: Suppose a gambler starts with an initial amount of money $\boldsymbol{b}$ and wishes to reach an amount of $\boldsymbol{a}$ before going broke (or, running out of $\boldsymbol{b}$.

(1) What is the probability that the gambler wins (reach $\boldsymbol{a}$ before losing all the money $\boldsymbol{b}$.

(2) What is the expected waiting time before losing all the money $\boldsymbol{b}$.

2 Presentation of (1)

Let $\boldsymbol{a}, \boldsymbol{b} > 0$ with $\boldsymbol{a} > \boldsymbol{b}$. Let $(\boldsymbol{B_t},\ \boldsymbol{t} \geq 0)$ be a standard Brownian motion starting at $\boldsymbol{B}_0 = \boldsymbol{b}$.
Define the first hitting time:

$$\tau(\boldsymbol{w}) = \inf(\boldsymbol{t} > 0 \ |\boldsymbol{B_t}(\boldsymbol{w}) \geq \boldsymbol{a} \text{ or } \boldsymbol{B_t}(\boldsymbol{w}) \leq 0).$$

The proof that $\tau(\boldsymbol{w}) < \infty$ except for a set of $\boldsymbol{w}$ of measure 0 is as follows: Consider an event that n-th increments exceeds $\boldsymbol{a}$, represented as

$$\boldsymbol{E}_n = \{|\boldsymbol{B_n} \text{ - } \boldsymbol{B_{n-1}}|\} > \boldsymbol{a}.$$

If $\boldsymbol{E}_n$ happens, indicating that the two consecutive gambling can give out a higher payoff than the sum of initial money and winning money for one time, then the Brownian path must exit $[0,\boldsymbol{a}]$.

Define probability that $\boldsymbol{E_n}$ happens as $\boldsymbol{P}(\boldsymbol{E_n})$. We have $\boldsymbol{P}(\boldsymbol{E_n}) = \boldsymbol{P}(\boldsymbol{E_1})$ $:=\boldsymbol{p}$ for all $\boldsymbol{n}$, with $\boldsymbol{p} \in (0, 1)$.
As the events $\boldsymbol{E_n}$ are independent, we have

$$\boldsymbol{P}(\boldsymbol{E}_1^{\boldsymbol{c}} \cap \boldsymbol{E}_2^{\boldsymbol{c}} \cap \boldsymbol{E}_3^{\boldsymbol{c}} ... \cap \boldsymbol{E}_n^{\boldsymbol{c}}) = \boldsymbol{P}(\boldsymbol{E}_1^{\boldsymbol{c}}) \boldsymbol{P}(\boldsymbol{E}_2^{\boldsymbol{c}}) \boldsymbol{P}(\boldsymbol{E}_3^{\boldsymbol{c}}) ... \boldsymbol{P}(\boldsymbol{E}_n^{\boldsymbol{c}}) = (1\text{-}\boldsymbol{p})^{\boldsymbol{n}}.$$

Following, we have

$$\lim_{n\to\infty} P(E_1^c \cap E_2^c \cap E_3^c ... \cap E_n^c) = \lim_{n\to\infty} (1\text{-}p)^n = 0$$

As such probability is equal to 0, E_n needs to have occurred with some n, thus $\tau(w) < \infty$.
If $\tau(w) = t$, then $B_t(w) = B_\tau(w)$, the question equals finding $P(B_\tau(w) = a)$.
We have

$$E(B_\tau) = aP(B_\tau = a) + 0(1 - P(B_\tau = a). \ (1)$$

From Theorem 3.2.18, we have $E(B_\tau) = E(B_0) = b$. (2)
From (1) and (2), we have

$$P(B_\tau = a) = \frac{b}{a}.$$

3 Presentation of (2)

Consider the same τ as described in the previous section.
Consider the martingale $M_t = B_t^2 - t$.
From Theorem 3.2.18, we also obtain

$E(M_\tau) = E(M_0) = E(B_0^2) - E(0) = b^2 1 - 0 = b^2.$

$$E(M_\tau) = E(B_\tau^2 - \tau) = E(B_\tau^2) - E(\tau).$$

From the previous section, we can infer
$E(B_\tau^2) = a^2 P(B_\tau = a) + 0^2(P(B\tau = 0) = a^2 \frac{b}{a} = ab.$

Combining every part, we obtain

$$b^2 = E(M_\tau) = ab - E(\tau)$$
$$\Leftrightarrow \quad E(\tau) = b(a - b).$$