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Stochastic: (Chapter 3: Conditional expectation and martingales)

Exercise 3.1.2: prove $\mathbb{E}(W\mathbb{E}(X|G)) = \mathbb{E}(WX)$.

· Using Definition 3.1.1: $(\Omega, F, \mathbb{P})$ be probability space, X a random variable on $(\Omega, F, \mathbb{P})$, taking with values in standard measurable space and belonging to $L^1(\Omega, F, \mathbb{P})$. Let G be a σ-subalgebra of F. The conditional expectation of X given G, denoted by $\mathbb{E}(X|G)$, is the random variable taking value in $(\Lambda, \mathcal{E})$, measurable with respect to G, belonging to $L^1(\Omega, G, \mathbb{P})$ and satisfying: $\int_D \mathbb{E}(X|G)\, d\mathbb{P} = \int_D X\, d\mathbb{P} \quad \forall D \in G$

$$\Leftrightarrow \mathbb{E}(\mathbb{E}(X|G) I_D) = \int_D \mathbb{E}(X|G)\, d\mathbb{P} = \int_D X\, d\mathbb{P} = \mathbb{E}(X.I_D) \quad \forall D \in G$$

· Using Proposition 1[1]: Let $(E, \mathcal{E})$ be a measurable space and f a positive real measurable function on E. There exists an increasing sequence $(f_n)_n$ of functions of the form:

$$f_n(x) = \sum_{i=1}^{n} \alpha_i I_{A_i}(x) \qquad \text{s.t. } f_n \nearrow f$$

· This imply that every measurable positive function is the increasing limit of a sequence of elementary functions.

· Using the linearity property of conditional expectation, we have:

$$\mathbb{E}(\mathbb{E}(X|G) I_D) = \mathbb{E}(X.I_D)$$

$$\Rightarrow \sum_n \alpha_k \mathbb{E}(\mathbb{E}(X|G) I_{D_k}) = \sum_n \alpha_k \mathbb{E}(X.I_{D_k})$$

· Let $W_n = \sum_{k=1}^{n} \alpha_k I_{D_k}$ s.t $W_n \nearrow W$. (W is positive)

[1]: P. Baldi, Stochastic calculus, an introduction through theory and exercises.

We have:

$$\mathbb{E}(W\,\mathbb{E}(X|G)) = \lim_{n\to\infty}\mathbb{E}(\mathbb{E}(X|G)\sum_n \alpha_k I_{D_k}) = \lim_{n\to\infty}\mathbb{E}(X.\sum_n \alpha_k I_{D_k})$$

$$= \mathbb{E}(X.W)$$

$\Rightarrow \mathbb{E}(W.X) = \mathbb{E}(W\,\mathbb{E}(X|G))$ $\forall$ bounded and G measurable univarate r.v. W on $(\Omega, F, \mathbb{P})$ (W is positive)