Stochastic (Chapter 2: Gaussian process)

Exercise 2.4.3: Show that the Brownian process is a Gaussian process

Proof:

- Definition of Gaussian process: The family $X := (X_t)_{t\in\tau}$ with each X_t a univariate random variable on the probability space $(\Omega, F, \mathbb{P})$ is a Gaussian process if for any finite family $\{t_1, t_2, \dots, t_n\} \subset \tau$ with $t_j < t_{j+1}$, the N-dimensional vector $(X_{t_1}, X_{t_2}, \dots, X_{t_N})^T$ is a Gaussian vector
- Definition of 1-dimensional Brownian motion: A stochastic process $B := (\Omega, F, \mathbb{P}, (F_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ taking values in R is a 1-dimensional Brownian motion if:

1. $B_0 = 0$ a.s
2. For any $0 \le s \le t$ the random variable $B_t - B_s$ is independent of F_s
3. For any $0 \le s \le t$ the random variable $B_t - B_s$ is a Gaussian random variable $N(0, t-s)$

- To prove the Brownian motion is a Gaussian process, we need to prove that $\alpha_1 B_{t_1} + \alpha_2 B_{t_2} + \dots + \alpha_m B_{t_m}$ is a r.v. Gaussian with $\alpha_1, \alpha_2, \dots, \alpha_m \in R$, $0 < t_1 < t_2 < \dots < t_m$.
- We are going to use the induction method to proof this exercise:

t_1 $m=1$: From 3, of the definition of B.M., $B_{t_1} - B_s$ is a Gaussian random variable $N(0, t-s)$. Let $s=0$, we have $B_{t_1} - B_0 = B_{t_1}$ is a Gaussian random variable

t_1 $m=2$: Let consider

$$\alpha_1 B_{t_1} + \alpha_2 B_{t_2} = (\alpha_1 + \alpha_2) B_{t_1} + \alpha_2 (B_{t_2} - B_{t_1})$$

We know that $(\alpha_1 + \alpha_2) B_{t_1}$ is a Gaussian random variable measurable with respect to F_{t_1}.

On the other hand, from 3, of the definition, we know that $\alpha_2 (B_{t_2} - B_{t_1})$ is a Gaussian random variable and from 2, of the definition, $\alpha_2 (B_{t_2} - B_{t_1})$ is independent of F_{t_1}.

- Using result of exercise 2.1.2[1]: Check that if X_1, X_2 are independent and standard Gaussian random variables, then $(X_1, X_2)^T$ is a Gaussian vector. Show that the random variable $a_1 X_1 + a_2 X_2$ is a Gaussian random variable with mean 0 and variance $a_1^2 + a_2^2$

$\rightarrow$ If X_1, X_2 are independent and standard Gaussian random variables, then $a_1 X_1 + a_2 X_2$ is a Gaussian random variable.

- With $(\alpha_1 + \alpha_2) B_{t_1}$ and $\alpha_2 (B_{t_2} - B_{t_1})$ are Gaussian random variables and independent with each other, we have

$$(\alpha_1 + \alpha_2) B_{t_1} + \alpha_2 (B_{t_2} - B_{t_1}) = \alpha_1 B_{t_1} + \alpha_2 B_{t_2}$$

is Gaussian random variable.

+, Assume that it is true for $m-1$

Let consider the $m = m$

$$\alpha_1 B_{t_1} + \alpha_2 B_{t_2} + \dots + \alpha_m B_{t_m} = (\alpha_1 B_{t_1} + \alpha_2 B_{t_2} + \dots + (\alpha_{m-1} + \alpha_m) B_{t_{m-1}}) + \alpha_m (B_{t_m} - B_{t_{m-1}})$$

We know that $(\alpha_1 B_{t_1} + \alpha_2 B_{t_2} + \dots + (\alpha_{m-1} + \alpha_m) B_{t_{m-1}})$ is a Gaussian random variable by the induction assumption and $F_{t_{m-1}}$ measurable. Using definition 3, of the B.M, we know that $\alpha_m (B_{t_m} - B_{t_{m-1}})$ is a Gaussian random variable and independent of $F_{t_{m-1}}$ by 2, from the definition.

Using the result of exercise 2.1.2[1]:

$\rightarrow (\alpha_1 B_{t_1} + \alpha_2 B_{t_2} + \dots + (\alpha_{m-1} + \alpha_m) B_{t_{m-1}}) + \alpha_m (B_{t_m} - B_{t_{m-1}}) = \alpha_1 B_{t_1} + \alpha_2 B_{t_2} + \dots + \alpha_m B_{t_m}$ is a Gaussian random variable

$\rightarrow$ Brownian process is a Gaussian process

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[1]: Sum of independent Gaussian random variables, Tetsuya Watari on the Stochastic website.