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Stochastic (Chapter 1: Mathematical Background)

Exercise 1.3.2: Check that the covariance matrix $Cov(X)$ is symmetric and positive semi-definite, namely it satisfies

$$a^T Cov(X)\, a \geq 0 \quad \forall a \in \mathbb{R}^N \text{ with } a \neq 0$$

Proof:

$$Cov(X) := Cov(X, X) \in M_{N \times N}(\mathbb{R})$$

$$\rightarrow cov(X) = \mathbb{E}\left((X - \mathbb{E}(X))(X - \mathbb{E}(X))^T\right)$$

$$\Leftrightarrow a^T Cov(X)\, a = a^T\, \mathbb{E}\left((X - \mathbb{E}(X))(X - \mathbb{E}(X))^T\right) a$$

$$\Leftrightarrow a^T Cov(X)\, a = \mathbb{E}\left(a^T (X - \mathbb{E}(X))(X - \mathbb{E}(X))^T a\right)$$

Let $A = a^T (X - \mathbb{E}(X)) = (X - \mathbb{E}(X))^T a$

$$\Rightarrow a^T Cov(X)\, a = \mathbb{E}(A^2)$$

From property of expected value, we have if $X \geq 0$, then $\mathbb{E}(X) \geq 0$

$$\rightarrow a^T Cov(X)\, a \geq 0$$

So $a^T Cov(X)\, a \geq 0 \quad \forall a \in \mathbb{R}^N$ with $a \neq 0$.

- Using $AB^T = (BA^T)^T$, we get:

$$(X - \mathbb{E}(X))(X - \mathbb{E}(X))^T = \left[(X - \mathbb{E}(X))(X - \mathbb{E}(X))^T\right]^T$$

$$\Leftrightarrow Cov(X) = \mathbb{E}\left((X - \mathbb{E}(X))(X - \mathbb{E}(X))^T\right) = \mathbb{E}\left[(X - \mathbb{E}(X))(X - \mathbb{E}(X))^T\right]^T = [Cov(X)]^T$$

$\rightarrow$ Covariance matrix is symmetric and positive semi-definite