Why normal distribution's integral is 1?

SML: Stochastic Calculus (Fall 23)
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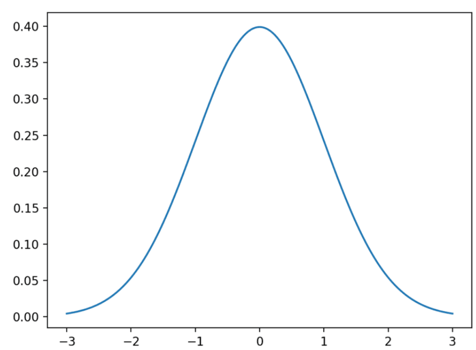

The Gaussian distribution. The most well-known distribution of continuous values of all. Also known as the *normal distribution* or the *bell curve*, we have taken for granted that its integral is 1, and just apply the formula. Let's take a look and see how to derive it.

Exercise 1.2.5. For $\sigma > 0$ and $\bar{x} \in \mathbb{R}$, set $\pi : \mathbb{R} \mapsto [0, \infty)$ by

$$\pi(x) := \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x - \bar{x})^2\right)$$

Check that $\int \pi(x)\,dx = 1$.

Proof. In this method, we shall introduce change-of-variable, particularly to apply *polar coordinates*. Let $w = \dfrac{x - \bar{x}}{\sigma}$ and $\int \pi(x)\,dx =: I$,

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x - \bar{x}}{\sigma}\right)^2\right) dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}w^2\right) dw. \end{aligned}$$

Square both side, we get

$$\begin{aligned}
I^2 &= \left(\int_{\infty}^{-\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx\right)\left(\int_{\infty}^{-\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy\right) \\
&= \frac{1}{2\pi} \iint \exp\left(-\frac{x^2}{2}\right) \exp\left(-\frac{y^2}{2}\right) dxdy \\
&= \frac{1}{2\pi} \iint \exp\left(-\frac{1}{2}\left(x^2 + y^2\right)\right) dxdy.
\end{aligned}$$

Using polar coordinates with $x = r\cos\theta$ and $y = r\sin\theta$ with adding the Jacobian, we obtain

$$I^2 = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^{\infty} \exp\left(-\frac{r^2}{2}\right) r\, dr\right) d\theta.$$

Another change-of-variables with $u = \dfrac{r^2}{2}$ and $du = rdr$, so now our integral becomes

$$\begin{aligned}
I^2 &= \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^{\infty} e^{-u}\, du\right) d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(e^{-u}\Big|_0^{\infty}\right) d\theta
\end{aligned}$$

$$\begin{aligned}
I^2 &= \frac{1}{2\pi} \int_0^{2\pi} -(0-1)\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} d\theta \\
&= \frac{1}{2\pi} (2\pi) \\
&= 1.
\end{aligned}$$

Since $I^2 = 1$, this implies $I = +1$ or $I = -1$. Thus, we need to show that in order for $I = \int \pi(x)\, dx = 1$, $\pi(x)$ is strictly positive.

This is easy. Looking at

$$\pi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(x - \bar{x})^2\right)$$

We know that standard deviation, σ is positive. $\sqrt{2\pi}$ is also positive. In addition, the natural exponential function is positive. Thus, multiplying the positive terms together will still yield positive terms only.

Therefore, $\pi(x) > 0$ and so,

$$\int \pi(x)\, dx = 1.$$

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