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3.1 (p. 455) Let B be a Brownian motion and let $s \le t$.

a) Compute $\mathrm{E}[B_s B_t^2]$.
b) Compute $\mathrm{E}[B_s^2 B_t^2]$.
c) Show that

$$\mathrm{E}[B_s e^{B_s}] = s e^{s/2} .$$

d) Compute $\mathrm{E}[B_s e^{B_t}]$.

Recall: $\mathrm{E}[X^4] = 3$ if $X \sim N(0,1)$.

[B] p.75

Remark 3.3 Computations concerning Brownian motion repeatedly require a certain set of formulas typical of Gaussian distributions. Let us recall them (they are all based on the relation $B_t \sim \sqrt{t} Z$ with $Z \sim N(0,1)$):

a) $\mathrm{E}[e^{\theta B_t}] = e^{\frac{1}{2} t\theta^2}$;

b) $\mathrm{E}[e^{\theta B_t^2}] = \begin{cases} \frac{1}{\sqrt{1-2t\theta}} & \text{if } t\theta < \frac{1}{2} \\ +\infty & \text{if } t\theta \ge \frac{1}{2} \end{cases}$.

[B]. p.52

Let's consider $\mathbb{E}[e^{\theta B_t}] = e^{1/2 t\theta^2}$ and show that $\mathbb{E}[B_t^n] = \frac{d^n}{d\theta^n} \mathbb{E}[e^{\theta B_t}]\Big|_{\theta=0}$

Using linear property of expectation,

* The expected value of the sum of random variables is equal to the sum of their individual expected values, regardless of whether they are independent

We get
$$\mathbb{E}[e^{\theta z}] = \mathbb{E}\left[1 + \frac{\theta z}{1!} + \frac{\theta^2 z^2}{2!} + \frac{\theta^3 z^3}{3!} + \dots\right] =$$
$$= 1 + \theta \mathbb{E}[z] + \frac{\theta^2}{2!}\mathbb{E}[z^2] + \frac{\theta^3}{3!}\mathbb{E}[z^3] + \dots$$
now this enables us to calculate any moment of random variable

$n=1$ $$\frac{d}{d\theta}\mathbb{E}[e^{\theta z}] = \mathbb{E}[z] + \theta\mathbb{E}[z^2] + \frac{\theta^2}{2!}\mathbb{E}[z^3] + \dots$$ then $\frac{d}{d\theta}\mathbb{E}[e^{\theta z}]\Big|_{\theta=0} = \mathbb{E}[z]$

$n=2$ $$\frac{d^2}{d\theta^2}\mathbb{E}[e^{\theta z}] = \mathbb{E}[z^2] + \theta\mathbb{E}[z^3] + \dots$$ then $\frac{d^2}{d\theta^2}\mathbb{E}[e^{\theta z}]\Big|_{\theta=0} = \mathbb{E}[z^2]$

$\vdots$

$$\frac{d^n}{d\theta^n}\mathbb{E}[e^{\theta z}]\Big|_{\theta=0} = \mathbb{E}[z^n] \Rightarrow \frac{d^n}{d\theta^n}\mathbb{E}[e^{\theta B_t}]\Big|_{\theta=0} = \mathbb{E}[B_t^n]$$

Then

$n=1$ $$\mathbb{E}[B_t] = \frac{d}{d\theta}\mathbb{E}[e^{\theta B_t}]\Big|_{\theta=0} = \frac{d}{d\theta}(e^{\frac{1}{2}t\theta^2})\Big|_{\theta=0} = e^{\frac{1}{2}t\theta^2}\,\theta t\Big|_{\theta=0} = 0$$

$n=2$ $$\mathbb{E}[B_t^2] = \frac{d^2}{d\theta^2}\mathbb{E}[e^{\theta B_t}]\Big|_{\theta=0} = \frac{d^2}{d\theta^2}(e^{\frac{1}{2}t\theta^2})\Big|_{\theta=0} = e^{\frac{1}{2}t\theta^2}(t + \theta^2 t^2)\Big|_{\theta=0} = t$$

$n=3$ $$\mathbb{E}[B_t^3] = \frac{d^3}{d\theta^3}\mathbb{E}[e^{\theta B_t}]\Big|_{\theta=0} = \frac{d^3}{d\theta^3}(e^{\frac{1}{2}t\theta^2})\Big|_{\theta=0} = e^{\frac{1}{2}t\theta^2}(3\theta t^2 + \theta^3 t^3)\Big|_{\theta=0} = 0$$

$n=4$ $$\mathbb{E}[B_t^3] = \frac{d^4}{d\theta^4}\mathbb{E}[e^{\theta B_t}]\Big|_{\theta=0} = \frac{d^4}{d\theta^4}(e^{\frac{1}{2}t\theta^2})\Big|_{\theta=0} = e^{\frac{1}{2}t\theta^2}(3t^2 + 6\theta^2 t^3 + \theta^4 t^4)\Big|_{\theta=0} = 3t^2$$

(1) $\mathbb{E}[B_t^{2k+1}] = \frac{d^{2k+1}}{d\theta^{2k+1}}\left(e^{\frac{1}{2}\theta^2 t}\right) = \frac{d^{2k+1}}{d\theta^{2k+1}}\left(1 + \frac{\left(\frac{t}{2}\right)\theta^2}{1!} + \frac{\left(\frac{t}{2}\right)^2\theta^{2\cdot 2}}{2!} + \dots + \frac{\left(\frac{t}{2}\right)^{k+1}\theta^{2\cdot(k+1)}}{(k+1)!} + \dots\right) =$

$$= \frac{\frac{(2k+2)!}{(2(k+1)-(2k+1))!}\left(\frac{t}{2}\right)^{k+1}\theta^{(2(k+1)-(2k+1))}}{(k+1)!} + \frac{\frac{(2k+4)!}{(2(k+2)-(2k+1))!}\left(\frac{t}{2}\right)^{k+2}\theta^{(2(k+2)-(2k+1))}}{(k+2)!} + \dots$$

$$\frac{d^{2k+1}}{d\theta^{2k+1}}\left(e^{\frac{1}{2}\theta^2 t}\right)\Big|_{\theta=0} = 0 + 0 + \dots + 0 = 0 \qquad \mathbb{E}[B_t^{2k+1}] = 0$$

(2) $\mathbb{E}[B_t^{2k}] = \frac{d^{2k}}{d\theta^{2k}}\mathbb{E}[e^{\theta B_t}] = \frac{d^{2k}}{d\theta^{2k}}\left(e^{\frac{1}{2}\theta^2 t}\right) = \frac{d^{2k}}{d\theta^{2k}}\left(1 + \frac{\left(\frac{t}{2}\right)\theta^2}{1!} + \frac{\left(\frac{t}{2}\right)^2\theta^4}{2!} + \dots + \frac{\left(\frac{t}{2}\right)^k\theta^{2k}}{k!} + \dots\right) =$

$$= \frac{(2k)!\left(\frac{t}{2}\right)^k\theta^{2k-2k}}{k!} + \frac{\frac{(2k+2)!}{(2(k+1)-2k)!}\left(\frac{t}{2}\right)^{k+1}\cdot\theta^{(2(k+1)-2k)}}{(k+1)!} + \dots$$

$$\frac{d^{2k}}{d\theta^{2k}}\mathbb{E}[e^{\theta B_t}]\Big|_{\theta=0} = \frac{(2k)!\left(\frac{t}{2}\right)^k}{k!} + 0 + 0 + \dots + 0 = \frac{(2k)!\,t^k}{2^k\cdot k!} =$$

$$= \frac{(2k)!\,t^k}{(2k)\cdot(2k-2)\cdot\ldots\cdot 2} = t^k(2k-1)\cdot(2k-3)\cdot\ldots\cdot 3\,1 = t^k(2k-1)!!$$

$$\mathbb{E}[B_t^{2k}] = t^k(2k-1)!!$$

From (1) and (2), we get the following.

When $k = 0, 1, 2, \dots$ $\quad \mathbb{E}[B_t^{2k+1}] = 0$ and $\mathbb{E}[B_t^{2k}] = t^k(2k-1)!!$

a.) $\mathbb{E}[B_s B_t^2] = \mathbb{E}[B_s(B_t - B_s + B_s)^2] = \mathbb{E}[B_s(B_t - B_s)^2 + 2B_s^2(B_t - B_s) + B_s^3] =$

$$= \mathbb{E}[B_s(B_t - B_s)^2] + 2\mathbb{E}[B_s^2(B_t - B_s)] + \mathbb{E}[B_s^3] =$$

$$= 0 + 0 + 0 = 0$$

b.) $\mathbb{E}[B_s^2 B_t^2] = \mathbb{E}[B_s^2(B_t - B_s + B_t)^2] = \mathbb{E}[B_s^2(B_t - B_s)^2 + 2B_s^3(B_t - B_s) + B_s^4] =$

$$= \mathbb{E}[B_s^2(B_t - B_s)^2] + \mathbb{E}[2B_s^3(B_t - B_s)] + \mathbb{E}[B_s^4] =$$

$$= s(t-s) + 0 + 3s^2 = st + 2s^2$$

c.) $$\mathbb{E}[B_s e^{B_s}] = \mathbb{E}\left[B_s\left(1+B_s+\frac{B_s^2}{2!}+\frac{B_s^3}{3!}+\dots\right)\right]$$

$$\boxed{\mathbb{E}[B_t^{2k+1}]=0} \Rightarrow = \underbrace{\mathbb{E}[B_s]}_{0}+\mathbb{E}[B_s^2]+\frac{1}{2!}\underbrace{\mathbb{E}[B_s^3]}_{0}+\frac{1}{3!}\mathbb{E}[B_s^4]+\dots$$

$$= \mathbb{E}[B_s^2]+\frac{1}{3!}\mathbb{E}[B_s^4]+\frac{1}{5!}\mathbb{E}[B_s^6]+\dots$$

$$\boxed{\mathbb{E}[B_t^{2k}]=t^k(2k-1)!!} \Rightarrow = s\cdot 1!!+\frac{1}{3!}s^2\cdot 3!!+\frac{1}{5!}s^3\cdot 5!!+\dots$$

$$= s+\frac{s^2}{2}+\frac{s^3}{4\cdot 2}+\frac{s^4}{6\cdot 4\cdot 2}+\dots$$

$$= s\left(1+\frac{\frac{s}{2}}{1}+\frac{\left(\frac{s}{2}\right)^2}{2\cdot 1}+\frac{\left(\frac{s}{2}\right)^3}{3\cdot 2\cdot 1}+\dots\right)$$

$$= s\,e^{\frac{s}{2}}$$

d.) $$\mathbb{E}[B_s e^{B_t}] = \mathbb{E}[B_s e^{B_s}\cdot e^{B_t-B_s}] \overset{*}{=} \mathbb{E}[B_s e^{B_s}]\,\mathbb{E}[e^{B_t-B_s}] =$$
$$= s\,e^{s/2}\cdot e^{(t-s)/2} = s\cdot e^{t/2}$$

* B_s and $B_t - B_s$ are independent

c) if z denotes an $N(0,1)$-distributed r.v, then $B_s \sim \sqrt{s}\,z$

$$\mathbb{E}[B_s e^{B_s}] = \mathbb{E}[\sqrt{s}\,z\,e^{\sqrt{s}z}] = \frac{\sqrt{s}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} z\,e^{\sqrt{s}z}\,e^{-z^2/2}\,dz =$$

$$= \underbrace{-\frac{\sqrt{s}}{\sqrt{2\pi}}\,e^{\sqrt{s}z}\,e^{-z^2/2}\Big|_{-\infty}^{+\infty}}_{0} + \frac{s}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{\sqrt{s}z}\,e^{-z^2/2}\,dz = s\,e^{s/2}$$