SML: Introduction to Stochastic Calculus
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On Martingales

Definition 3.2.1 (Martingale, supermartingale, submartingale). *For $\mathcal{T} \subset \mathbb{R}_+$, a real valued stochastic process $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathcal{T}}, (M_t)_{t\in\mathcal{T}})$ satisfying $M_t \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ for any $t \in \mathcal{T}$ is a* martingale *if $\mathbb{E}(M_t|\mathcal{F}_s) = M_s$ for all $s \leq t$. It is a* supermartingale *if $\mathbb{E}(M_t|\mathcal{F}_s) \leq M_s$ or a* submartingale *if $\mathbb{E}(M_t|\mathcal{F}_s) \geq M_s$.*

Exercise 3.2.2. *Let $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let X be a univariate random variable on this space, with $\mathbb{E}(|X|) < \infty$. Set $X_t := \mathbb{E}(X|\mathcal{F}_t)$. Show that $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathcal{T}}, (X_t)_{t\in\mathcal{T}})$ is a martingale.*

$$\mathbb{E}(|X|) < \infty \Rightarrow X \in \mathcal{L}^1\{\Omega, \mathcal{F}, \mathbb{P}\} \quad [\text{Def } 1.4.1]$$

set $X_t := \mathbb{E}(X | \mathcal{F}_t)$

then,

$$\mathbb{E}(X_t | \mathcal{F}_s) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_t) | \mathcal{F}_s) \quad \text{for } [s<t]$$

$$\downarrow \text{prop } 3.13\ (5)$$

$$= \mathbb{E}(X | \mathcal{F}_s)$$

$$= X_s \quad \blacksquare$$

Therefore X_t is a martingale.

Exercise 3.2.4. *Show that the standard*[5] *1-dimensional Brownian process is a martingale.*

Let $B := (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ taking values in $\mathbb{R}$ be a 1-D Brownian motion. Then,

$$\mathbb{E}(B_t | \mathcal{F}_s) = \mathbb{E}(B_t - B_s + B_s | \mathcal{F}_s)$$

$$= \mathbb{E}(B_t - B_s | \mathcal{F}_s) + \mathbb{E}(B_s | \mathcal{F}_s)$$

$$\downarrow ①$$

$$= \mathbb{E}(B_t - B_s) + \mathbb{E}(B_s | \mathcal{F}_s)$$

$$= 0 + B_s$$

①: $B_t - B_s$ and B_s are independent by definion.

$\therefore$ 1D Brownian motion is a martingale.

Exercise 3.2.6. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ be the standard 1-dimensional Brownian process. Show that the new process defined by $X_t := B_t^2$ is a submartingale, but that the process defined by $X_t := B_t^2 - t$ is a martingale.*

To prove $X_t := B_t^2 - t$ is a martingale :

$$\mathbb{E}(X_t \mid \mathcal{F}_s) = \mathbb{E}(B_t^2 - t \mid \mathcal{F}_s) \qquad [\text{for } s < t]$$

$$= \mathbb{E}((B_t - B_s + B_s)^2 - t \mid \mathcal{F}_s)$$

$$= \mathbb{E}((B_t - B_s)^2 + 2(B_t - B_s)B_s + B_s^2 - t \mid \mathcal{F}_s)$$

$$= \mathbb{E}((B_t - B_s)^2 \mid \mathcal{F}_s) + 2\mathbb{E}((B_t - B_s)B_s \mid \mathcal{F}_s) + \mathbb{E}(B_s^2 - t \mid \mathcal{F}_s)$$

$$= \mathbb{E}((B_t - B_s)^2) + 2\underbrace{\cancel{\mathbb{E}}(B_s)}_{0}\mathbb{E}(B_t - B_s) + B_s^2 - t$$

$$= (t - s) + B_s^2 - t$$

$$= B_s^2 - s \qquad \blacksquare$$

To prove that $X_t := B_t^2$ is a submartingale. For $s < t$ we have:

$$\mathbb{E}(X_t \mid \mathcal{F}_s) = \mathbb{E}(B_t^2 \mid \mathcal{F}_s)$$

$$= \mathbb{E}((B_t - B_s + B_s)^2 \mid \mathcal{F}_s)$$

$$= \mathbb{E}((B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 \mid \mathcal{F}_s)$$

$$= \mathbb{E}((B_t - B_s)^2 \mid \mathcal{F}_s) + 2\mathbb{E}(B_s(B_t - B_s) \mid \mathcal{F}_s) + \mathbb{E}(B_s^2 \mid \mathcal{F}_s)$$

$$= (t - s) + 2\underbrace{\cancel{\mathbb{E}}(B_s)}_{0}\mathbb{E}(B_t - B_s) + B_s^2$$

$$= (t - s) + B_s^2 \geqslant B_s^2$$

$\therefore X_t := B_t^2$ is a submartingale.

Exercise 3.2.5. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ be the standard 1-dimensional Brownian process, and consider the* geometric Brownian process *defined by $S_t := S_0 \exp(\sigma B_t + \mu t)$, with $\sigma > 0$, $\mu \in \mathbb{R}$, and $S_0 \in \mathbb{R}$ an arbitrary initial value. Show that this process is a martingale if and only if $\mu = -\frac{1}{2}\sigma^2$.*

A stochastic process $(X_t)_{t\in\mathcal{T}}$ adapted to the filteration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ is a martingale if for all $s<t$, $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$.

Given,

$$S_t = S_0 \exp(\sigma B_t + \mu t) \qquad [\sigma > 0 \ \& \ \mu \in \mathbb{R}, S_0 \in \mathbb{R}]$$

Now,

$$\mathbb{E}(S_t|\mathcal{F}_s) = \mathbb{E}\left(S_0 \exp(\sigma B_t + \mu t) \middle| \mathcal{F}_s\right)$$

$$= \mathbb{E}\left(S_0 \exp(\sigma(B_s + (B_t - B_s)) + \mu(s + (t-s)) \middle| \mathcal{F}_s\right)$$

$$= \mathbb{E}\left(S_0 \exp(\sigma B_s + \mu s) \exp(\sigma(B_t - B_s) + \mu(t-s)) \middle| \mathcal{F}_s\right)$$

$$\text{(A)} \quad = S_0 \exp(\sigma B_s + \mu s)\, \mathbb{E}\left(\exp(\sigma(B_t - B_s) + \mu(t-s)) \middle| \mathcal{F}_s\right)$$

Now if $(S_t)_{t\in\mathcal{T}}$ is a martingale, then $\mathbb{E}(S_t|\mathcal{F}_s) = S_s$

$$\Rightarrow \text{(A)} = S_0 \exp(\sigma B_s + \mu s)$$

$$\Rightarrow \mathbb{E}\left(\exp(\sigma(B_t - B_s) + \mu(t-s)) \middle| \mathcal{F}_s\right) = 1$$

Since $B_t - B_s$ is independent of $\mathcal{F}_s$ we have:

$$= \mathbb{E}\left(\exp(\sigma(B_t - B_s) + \mu(t-s))\right) = 1$$

We know the mgf for a 1D Gaussian process is given by:

$$M_Z(t) = \exp\left(\bar{x}t + \frac{1}{2}\sigma^2 t^2\right)$$

We know that the increment $B_t - B_s$ of a Brownian motion is normally distributed with mean 0 and variance $t-s$, i.e $B_t - B_s \sim N(0, t-s)$. Therefore the mgf for $\sigma(B_t - B_s)$ is given by:

$$\mathbb{E}(\exp(\sigma(B_t - B_s))) = \exp\left(\frac{1}{2}\sigma^2(t-s)\right)$$

(follows from prev page)

$$\Rightarrow \exp\left(\mu(t-s) + \frac{1}{2}\sigma^2(t-s) \right) = 1$$

$$\mu(t-s) + \frac{1}{2}\sigma^2(t-s) = 0$$

$$\Rightarrow \boxed{\mu = -\frac{1}{2}\sigma^2}$$

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