SML: Introduction to Stochastic Calculus
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Submitted by:
Pratham
Dhomne
(062201880)

On conditional expectation and Lp -spaces

Exercise 3.1.5 (♥). *In the framework of the previous proposition and for univariate random variables, show that the map $X \mapsto \mathbb{E}(X|\mathcal{G})$ is a bounded linear map from $L^p(\Omega, \mathcal{F}, \mathbb{P})$ to $L^p(\Omega, \mathcal{G}, \mathbb{P})$ with a norm smaller or equal to 1, for any $p \geq 1$. More explicitly, show the linearity and that $\mathbb{E}(|\mathbb{E}(X|\mathcal{G})|^p) \leq \mathbb{E}(|X|^p)$. In the proof, use Jensen's inequality for the function $x \mapsto |x|^p$.*

Recalling the proposition:

Proposition 3.1.3. *Let X, X^1, X^2 be random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a standard measurable space $(\Lambda, \mathcal{E})$, and assume that these random variables belong to $L^1(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{G}$ be a σ-subalgebra of $\mathcal{F}$, and let $\alpha, \beta \in \mathbb{R}$.*

1. $\mathbb{E}(\alpha X^1 + \beta X^2|\mathcal{G}) = \mathbb{E}(\alpha X^1|\mathcal{G}) + \mathbb{E}(\beta X^2|\mathcal{G})$,
2. *If X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$,*
3. *If $X \geq 0$ a.s., then $\mathbb{E}(X|\mathcal{G}) \geq 0$ a.s.,*
4. *If W is an univariate bounded and $\mathcal{G}$-measurable random variable, then $\mathbb{E}(WX|\mathcal{G}) = W\mathbb{E}(X|\mathcal{G})$ a.s.,*
5. *If $\mathcal{G}'$ is another σ-subalgebra of $\mathcal{F}$ satisfying $\mathcal{G} \subset \mathcal{G}'$, then $\mathbb{E}(\mathbb{E}(X|\mathcal{G}')|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})$ a.s.,*
6. *If X is independent of $\mathcal{G}$, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s., where $\mathbb{E}(X)$ can be considered as a constant random variable,*
7. *If X is univariate and $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex lower semi-continuous function, then*

$$\mathbb{E}(\varphi(X)|\mathcal{G}) \geq \varphi(\mathbb{E}(X|\mathcal{G})). \qquad \text{(Jensen's inequality)}$$

For a random variable X s.t $X \in L^p(\Omega, \mathcal{F}, P)$ we have to show that the map $X \to \mathbb{E}(X|G)$ is a bounded linear map from $L^p(\Omega, \mathcal{F}, P) \to L^p(\Omega, G, P)$.

(i) Linearity:

Consider two r.v $X, Y \in L^p(\Omega, \mathcal{F}, P)$ and scalars $a, b \in \mathbb{R}$ then,

$$(aX+bY) \to \mathbb{E}(aX+bY|G) = a\,\mathbb{E}(X|G) + b\,\mathbb{E}(Y|G)$$ [By linearity of $\mathbb{E}$, see prop 3.1.3 (1)]

i.e $(aX+bY) \longrightarrow a\,\mathbb{E}(X|G) + b\,\mathbb{E}(Y|G)$ ∎

(ii) Boundedness :

Lets first look at definitions related to a convex lower semi-continous function.

Def [convex function] A function $f: X \to \mathbb{R}$ is called convex if, for all $0 \leq t \leq 1$ and all $x_1, x_2 \in X$:

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$$

Coming back to the exercise, the function $\varphi(x) = |x|^p$ satisfies the above conditions for $p \geq 1$, hence is convex and is continous.

$$\varphi : x \to |x|^p$$

Then by Jensen's ineq :

$$\mathbb{E}(\varphi(X) | G) \geq \varphi(\mathbb{E}(X|G))$$

$$\Rightarrow \mathbb{E}(|X|^p | G) \geq |\mathbb{E}(X|G)|^p$$

Taking $\mathbb{E}$ both sides

$$\Rightarrow \mathbb{E}(\mathbb{E}(|X|^p|G)) \geq \mathbb{E}(|\mathbb{E}(X|G)|^p)$$

$$\Leftrightarrow \mathbb{E}(|X|^p) \geq \mathbb{E}(|\mathbb{E}(X|G)|^p)$$

Taking pth root both sides

$$\Leftrightarrow [\mathbb{E}(|X|^p)]^{1/p} \geq [\mathbb{E}(|\mathbb{E}(X|G)|^p)]^{1/p}$$

$$\Leftrightarrow \|X\|_p \geq \|\mathbb{E}(X|G)\|_p \qquad \blacksquare$$

Therefore,

$X \to \mathbb{E}(X|G)$ is a bounded linear map from $L^p(\Omega, \mathcal{F}, \mathbb{P})$ to $L^p(\Omega, G, \mathbb{P})$ with a norm smaller than or equal to 1 for any $p \geq 1$.