Nagoya University G30

SML: Introduction to stochastic calculus
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On Gaussian Random Variables and the Gaussian Integral

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This report aims to provide a solution to the Exercise **1.2.5.** of the lecture notes. We first start with a short overview on the Gaussian integral which we would encounter while evaluating several integrals for the statements discussed in the exercise.

Contents

1 The Gaussian integral

The Gaussian function comes up quite often in the study of probability and statistics, because of its relation to the normal distribution. If we restrict ourselves to one dimension, the standard Gaussian function is defined as $f(x) = e^{-x^2}$. If we take an indefinite integral of the standard Gaussian function, it can be shown that it cannot be expressed in terms of elementary functions. Interestingly enough, it is possible to evaluate the definite integral of the standard Gaussian function over the real line. In fact, for any $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$:

$$\int_{-\infty}^{\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}}.$$

For the sake of simplicity, we make use of multivariable calculus and polar coordinates, transforming the integral into a form that is easier to compute. Precisely, we make use of the following property:

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy$$

The polar coordinate transformation simplifies the integration process, allowing for a more manageable computation of the Gaussian integral, in the following way: Set $x^2 + y^2 = r^2$ and switching to polar coordinates which introduces a factor or r since the Jacobian of this transformation is r:

$$\begin{aligned}
\iint_{\mathbb{R}^2} e^{-(x^2+y^2)} dxdy &= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2} r dr d\theta \\
&= \int_{\theta=0}^{2\pi} d\theta \int_{0}^{\infty} r e^{-r^2} dr \qquad \text{(set } -r^2 = s\text{)} \\
&= 2\pi \int_{-\infty}^{0} \frac{1}{2} e^{s} ds \\
&= \pi \int_{-\infty}^{0} e^{s} ds \\
&= \pi \left(e^0 - e^{-\infty}\right) \\
&= \pi.
\end{aligned}$$

Therefore we have:

$$\left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \pi$$

i.e.,

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}.$$

Similarly, one can show that

$$\int_{\mathbb{R}} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}}. \tag{1}$$

2 The Univariate Gaussian Random Variable

Recall that any function $\Pi : \mathbb{R}^N \to [0, \infty)$ satisfying $\int \Pi(x)dx = 1$, or any function p from a finite set or a countable set Λ to $[0, 1]$ satisfying $\sum_x p(x) = 1$, defines the *pdf* or the *pmf* of a random variable.

Exercise 1.2.5

For $\sigma > 0$ and $\overline{x} \in \mathbb{R}$, set $\Pi : \mathbb{R} \to [0, \infty)$ by:

$$\Pi(x) := \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x - \overline{x})^2\right)$$

Check that $\int_{\mathbb{R}} \Pi(x)dx = 1$. In the framework of Reminder **1.1.12**, we write $X = N(\overline{x}, \sigma^2)$ for the corresponding univariate random variable, called a Gaussian random variable. Check that $\mathbb{E}(X) = \overline{x}$ and $Var(X) = \sigma^2$.

Hence, we start by showing $\int_{\mathbb{R}} \Pi(x)dx = 1$:

$$\int_{\mathbb{R}} \Pi(x)dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx \ .$$

We make the following substitution,

$$u = \sqrt{\frac{1}{2\sigma^2}}(x-\overline{x}).$$

Note that the Jacobian of this transformation is $\sqrt{2}\sigma$, hence we have:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx &= \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} e^{-u^2}\sqrt{2}\sigma du \\
&= \frac{1}{\sqrt{2\pi}\sigma}\sqrt{2\pi}\sigma \qquad \text{(from (1))} \\
&= 1 \ .
\end{aligned}$$

2.1 Expectation of X or $\mathbb{E}\left(N(\overline{x}, \sigma^2)\right)$

Lets start by recalling the definition of Expectation:

Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(\Lambda, \mathcal{E})$ and $(\Xi, \mathcal{G})$ be measurable spaces and assume $(\Xi, \mathcal{G})$ be standard, let $X : \Omega \to \Lambda$ be a random variable, and let $f : \Lambda \to \Xi$ be a measurable function. The expectation of $f(X)$ is defined by:

$$\mathbb{E}(f(X)) := \int_{\Lambda} f(x)\mu_x(dx)$$

If $(\Lambda, \mathcal{E})$ is standard and equal to $(\Xi, \mathcal{G})$ and if f denotes the indentity function id with $id(x) = x$, then we write $\mathbb{E}(X)$ for $\mathbb{E}(id(X))$, and call it the mean value of X, of the expectation of X.

Note that in the current setting, $(\Lambda, \mathcal{E})$ and $(\Xi, \mathcal{G})$ both are standard i.e. $(\mathbb{R}, \sigma_B)$. Hence, we are in the setting of a regular Riemann integration and have:

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) \cdot x dx \\
&= \frac{1}{\sqrt{2\pi}\sigma}\left(\int_{-\infty}^{\infty}(x-\overline{x}) \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx + \int_{-\infty}^{\infty} \overline{x} \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx \right) \ .
\end{aligned}$$

Looking at the first integral:

$$\int_{-\infty}^{\infty}(x-\overline{x}) \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx \ .$$

We make the following substitution $u = \frac{1}{\sqrt{2}\sigma}(x - \overline{x})$, the Jacobian of this transformation is $\sqrt{2}\sigma$ and the bounds of integration remain the same. Hence we have,

$$\begin{aligned}\int_{\mathbb{R}} (x-\overline{x}) \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx &= \int_{-\infty}^{\infty} \sqrt{2}\sigma u \exp\left(-u^2\right)\sqrt{2}\sigma du \\ &= 2\sigma^2 \int_{-\infty}^{\infty} u \exp\left(-u^2\right) du \ .\end{aligned}$$

Notice that $u \exp(-u^2)$ is an odd function, hence its integral over a symmetric domain should be zero. Hence we have,

$$\begin{aligned}\mathbb{E}(X) &= 0 + \frac{1}{\sqrt{2\pi}\sigma}\left(\int_{\mathbb{R}} \overline{x} \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx\right) \\ &= \frac{\overline{x}}{\sqrt{2\pi}\sigma}\left(\int_{\mathbb{R}} \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx\right) \\ &= \frac{\overline{x}}{\sqrt{2\pi}\sigma} \cdot \sqrt{2\pi}\sigma && \text{(from (1))} \\ &= \overline{x} \ .\end{aligned}$$

Therefore, we have shown that $\mathbb{E}(X) = \overline{x}$.

2.2 Variance of X or Var($N(\overline{x}, \sigma^2)$)

Variance

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let X be a univariate random variable defined on it. Then the variance of X is defined by

$$\mathrm{Var}(X) \coloneqq \mathbb{E}\left((X - \mathbb{E}(X))^2\right)$$

Hence we have,

$$\mathrm{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}((X - \overline{x})^2)$$

That is,

$$\begin{aligned}\mathrm{Var}(X) &= \int_{-\infty}^{\infty} \frac{(x-\overline{x})^2}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx = \int_{-\infty}^{\infty} \frac{(x^2 - 2x\overline{x} + \overline{x}^2)}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx \\ &= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} x^2 \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx - \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} 2x\overline{x} \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx \\ &\quad + \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \overline{x}^2 \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx \\ &= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx - \overline{x}^2 \ .\end{aligned}$$

Hence, we just have to evaluate the first integral. For which we can make the following substitution: $u = x - \overline{x}$

$$\int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\overline{x})^2\right) dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{\mathbb{R}} (u+\overline{x})^2 \exp\left(-\frac{u^2}{2\sigma^2}\right) du$$

Expanding the first term inside the integral we get,

$$\frac{1}{\sqrt{2\pi}\sigma}\left(\int_{-\infty}^{\infty} u^2 \exp\left(-\frac{u^2}{2\sigma^2}\right) + 2\overline{x}\int_{-\infty}^{\infty} u \exp\left(-\frac{u^2}{2\sigma^2}\right) + \overline{x}^2\int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2\sigma^2}\right)\right)$$

Now, notice that the third integral is trivial and is just unity when multiplied by the constant term in the front. The first 2 integrals on the other hand, can be evaluated using integration by parts and it can be checked that we get the following result,

$$\frac{1}{\sqrt{2\pi}\sigma}\left(\sigma^3 \cdot \sqrt{2\pi} \ + 0 + \ \overline{x}^2 \cdot \sqrt{2\pi}\sigma\right) = \sigma^2 + \overline{x}^2$$

The result $\mathbb{E}(X^2) = \sigma^2 + \overline{x}^2$ can also be taken as a property for the Gaussian random variable. Therefore we have,

$$\mathrm{Var}(X) = (\sigma^2 + \overline{x}^2) - \overline{x}^2 = \sigma^2$$

Summarizing, we have shown the following for the univariate Gaussian random variable X:

1. $\int_{\mathbb{R}} \Pi(x)dx = 1$
2. $\mathbb{E}(X) = \overline{x}$
3. $\mathrm{Var}(X) = \sigma^2$

3 The Multivariate Gaussian Random Variable

In this section we consider the N dimensional multivariate Gaussian random variable. We first begin with providing a proof that the *p.d.f* is normalized.

Multivariate Gaussian Random Variable

For $\overline{\mathbf{x}} \in \mathbb{R}^N$ and $P \in M_{N\times N}(\mathbb{R})$ with $P > 0$, set $\Pi : \mathbb{R}^N \to \mathbb{R}_+$ with

$$\Pi(\mathbf{x}) := \frac{1}{\sqrt{(2\pi)^N|P|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\overline{\mathbf{x}})^T P^{-1}(\mathbf{x}-\overline{\mathbf{x}})\right) \tag{2}$$

Where $P > 0$ meaning P is positive definite $\iff \mathbf{y}^T P\mathbf{y} > 0$ for all $\mathbf{y} \in \mathbb{R}^N\backslash\{0\}$ and $|P| := \det(P)$. We write $\mathbf{X} = N(\overline{\mathbf{x}}, P)$ for the corresponding multivariate random variable, called the N-dim Gaussian random variable.

To show :

$$\int_{\mathbb{R}^N} \Pi(\mathbf{x})d\mathbf{x} = 1 \iff \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2}(\mathbf{x}-\overline{\mathbf{x}})^T P^{-1}(\mathbf{x}-\overline{\mathbf{x}})\right)d\mathbf{x} = \sqrt{(2\pi)^N|P|}$$

We first consider the exponential part

$$\exp\left(-\frac{1}{2}(\mathbf{x}-\overline{\mathbf{x}})^T P^{-1}(\mathbf{x}-\overline{\mathbf{x}})\right) .$$

For simplicity lets make the following change of variables $\mathbf{y} = (\mathbf{x} - \bar{\mathbf{x}})$. Note that the Jacobain of this transformation is 1 so our integral remains the same. Hence we can rewrite the exponential as following:

$$\exp\left(-\frac{1}{2}(\mathbf{x} - \bar{\mathbf{x}})^T P^{-1}(\mathbf{x} - \bar{\mathbf{x}})\right) = \exp\left(-\frac{1}{2}\mathbf{y}^T P^{-1}\mathbf{y}\right).$$

Going back to the assumptions on the matrix P, we observe that since $P > 0$ (i.e. P is positive definite) all of its eigenvalues are strictly positive and hence we can diagonalize it. Precisely, there exists an *orthogonal* matrix U and a diagonal matrix S made up of eigenvalues of P, such that $P = USU^T$. Therefore, $P^{-1} = US^{-1}U^T$ substituting this into the exponential to get:

$$\exp\left(-\frac{1}{2}\mathbf{y}^T US^{-1}U^T\mathbf{y}\right) = \exp\left(-\frac{1}{2}(U^T\mathbf{y})^T S^{-1}U^T\mathbf{y}\right).$$

Now we make another substitution $\mathbf{z} = U^T\mathbf{y}$, since U is an orthogonal matrix we have $\det(U) = \det(U^T) = 1$ therefore the Jacobian of this transformation is also 1, hence our integral does not change.

$$\exp\left(-\frac{1}{2}(U^T\mathbf{y})^T S^{-1}U^T\mathbf{y}\right) = \exp\left(-\frac{1}{2}\mathbf{z}^T S^{-1}\mathbf{z}\right).$$

Now lets consider the diagonal matrix S made up of all the eigenvalues of P. Let the set of eigenvalues of P be $\{\lambda_i\}_{i=1}^N$ where $\lambda_i \in \mathbb{R}$ and $\lambda_i > 0\ \forall i \in \{1, .., N\}$.

$$S = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix} \implies S^{-1} = \begin{pmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\lambda_N} \end{pmatrix}.$$

So the product $\mathbf{z}^T S^{-1}\mathbf{z}$ reduces down just a sum of squares, precisely

$$\mathbf{z}^T S^{-1}\mathbf{z} = \sum_{i=1}^{N} \frac{z_i^2}{\lambda_i}.$$

Substituting this in the exponential it becomes,

$$\exp\left(-\frac{1}{2}\mathbf{z}^T S^{-1}\mathbf{z}\right) = \exp\left(-\frac{1}{2}\sum_{i=1}^{N}\frac{z_i^2}{\lambda_i}\right) = \prod_{i=1}^{N}\exp\left(-\frac{z_i^2}{2\lambda_i}\right)$$

Hence, our original integral now becomes

$$\begin{aligned}\int_{\mathbb{R}^N} \exp\left(-\frac{1}{2}(\mathbf{x} - \bar{\mathbf{x}})^T P^{-1}(\mathbf{x} - \bar{\mathbf{x}})\right) d\mathbf{x} &= \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2}\mathbf{z}^T S^{-1}\mathbf{z}\right) d\mathbf{z} \\ &= \int_{\mathbb{R}^N} \prod_{i=1}^{N}\exp\left(-\frac{z_i^2}{2\lambda_i}\right) d\mathbf{z} \\ &= \prod_{i=1}^{N}\int_{-\infty}^{\infty}\exp\left(-\frac{z_i^2}{2\lambda_i}\right) dz_i\end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{N} \sqrt{2\pi\lambda_i} \\
&= \sqrt{(2\pi)^n |\lambda_1 \cdot \lambda_2 \cdots \lambda_N|} \\
&= \sqrt{(2\pi)^N |P|} \ . \quad \square
\end{aligned}
$$

3.1 Expectation of X or $\mathbb{E}\left(N(\overline{\mathbf{x}}, P)\right)$

The expectation of a random variable $\mathbf{X}$ as discussed in the univariate case is given by the following expression

$$
\mathbb{E}(f(\mathbf{X})) = \int_{\Lambda} f(x)\mu_{\mathbf{X}}(dx)
$$

Hence the expectation of the multivariate gaussian random variable can be written as:

$$
\mathbb{E}(\mathbf{X}) = \int_{\mathbb{R}^N} \frac{\mathbf{x}}{\sqrt{(2\pi)^N |P|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\overline{\mathbf{x}})^T P^{-1}(\mathbf{x}-\overline{\mathbf{x}})\right) d\mathbf{x} \ .
$$

One notable difference from the univariate case is that the left hand side of this equation $\mathbb{E}(\mathbf{X})$ is itself a N-dim vector, we will have an integral for each component x_i of $\mathbf{x}$. Proceeding with a similar fashion as the univariate case we can rewrite the above integral as

$$
\begin{aligned}
\mathbb{E}(\mathbf{X}) = &\int_{\mathbb{R}^N} \frac{\mathbf{x}-\overline{\mathbf{x}}}{\sqrt{(2\pi)^N |P|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\overline{\mathbf{x}})^T P^{-1}(\mathbf{x}-\overline{\mathbf{x}})\right) d\mathbf{x} \\
&+ \int_{\mathbb{R}^N} \frac{\overline{\mathbf{x}}}{\sqrt{(2\pi)^N |P|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\overline{\mathbf{x}})^T P^{-1}(\mathbf{x}-\overline{\mathbf{x}})\right) d\mathbf{x} \ .
\end{aligned}
$$

With suitable substitutions it can be shown that first integral vanishes and we only have the second integral.

$$
\begin{aligned}
\mathbb{E}(\mathbf{X}) &= \int_{\mathbb{R}^N} \frac{\overline{\mathbf{x}}}{\sqrt{(2\pi)^N |P|}} \exp\left(-\frac{1}{2}(\mathbf{x}-\overline{\mathbf{x}})^T P^{-1}(\mathbf{x}-\overline{\mathbf{x}})\right) d\mathbf{x} \\
&= \frac{\overline{\mathbf{x}}}{\sqrt{(2\pi)^N |P|}} \int_{\mathbb{R}^N} \exp\left(-\frac{1}{2}(\mathbf{x}-\overline{\mathbf{x}})^T P^{-1}(\mathbf{x}-\overline{\mathbf{x}})\right) d\mathbf{x} \\
&= \frac{\overline{\mathbf{x}}}{\sqrt{(2\pi)^N |P|}} \cdot \sqrt{(2\pi)^N |P|} \\
&= \overline{\mathbf{x}} \ .
\end{aligned}
$$