On the MGF of a Gaussian vector

SML Fall 2023: Introduction to Stochastic Calculus

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1 Motivation

The aim of this report is to provide an equivalent definition for a Gaussian vector via the Moment Generating Function (MGF). The main idea can be found in [Arg] Proposition 2.9, as the proof is based on the definition of a Gaussian vector, as well as some properties of the expectation, variance, and MGF.

2 Background

We shall start by recalling the definition of a Gaussian random variable and a Gaussian vector.

Definition 1. (Gaussian random variable) *For $\sigma > 0$ and $\bar{x} \in \mathbb{R}$ set $\Pi : \mathbb{R} \to \mathbb{R}_+$ by*

$$\Pi(x) := \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x - \bar{x})^2\right),$$

we write $X = N(\bar{x}, \sigma^2)$ for the corresponding univariate random variable, called Gaussian random variable. *One may check that $\mathbb{E}(X) = \bar{x}$, and $\mathrm{Var}(X) = \sigma^2$.*

Definition 2. (Gaussian vector) *An N-dimensional random vector $X = (X_1, ..., X_N)^T$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a* Gaussian vector *if for any $a = (a_1, ..., a_N)^T \in \mathbb{R}^N$ $(a \neq 0)$ the random variable $a \cdot X := \sum_{j=1}^{N} a_j X_j$ is a Gaussian random variable on $(\Omega, \mathcal{F}, \mathbb{P})$.*

Definition 3. *For $\bar{x} \in \mathbb{R}^N$ and $P \in M_{N\times N}(\mathbb{R})$ with $P > 0$, set $\Pi : \mathbb{R}^N \to \mathbb{R}_+$ by*

$$\Pi(x) := \frac{1}{(2\pi)^{N/2}|P|^{1/2}} \exp\left(-\frac{1}{2}(x - \bar{x})^T P^{-1}(x - \bar{x})\right),$$

*with $|P| := \det(P)$. We write $X = N(\bar{x}, P)$ for the corresponding multivariate random variable, called N-*dimensional Gaussian random variable. *One may check that $\mathbb{E}(X) = \bar{x}$, and that $P = \mathbb{E}((X - \bar{x})(X - \bar{x})^T)$. Here, P is called the covariance matrix, for which we write* $\mathrm{Cov}(X) := P$.

Remark: *The function Π for both Gaussian random variables is called the probability density function, or simply the* PDF.

3 An Alternative View on Gaussian Vectors

The goal of this report is to prove the following proposition, corresponding to Proposition 2.1.4. from [Ric], subsequently giving an alternative definition for a Gaussian vector.

Proposition 4 *An N-dimensional random vector $X = (X_1, ..., X_N)^T$ is Gaussian if and only if its moment generating function $\mathbb{E}(e^{a \cdot X})$ exists for all $a \in \mathbb{R}^N$ $(a \neq 0)$ and satisfies*

$$\mathbb{E}(e^{a \cdot X}) = \exp\Big(a \cdot \mathbb{E}(X) + \tfrac{1}{2} a^T \mathrm{Cov}(X) a\Big). \tag{1}$$

Proof. *Consider "only if" direction:* Given X a Gaussian random vector, we want to show that its MGF will have the form of (1). By definition 2, $a \cdot X$ is a univariate Gaussian random variable for all $a \in \mathbb{R}^N$ $(a \neq 0)$. We shall write Y for the corresponding univariate Gaussian random variable.

The following proof is similar to Gaussian vector of standard Gaussian distributions by Uyanga Khoroldagva but covers more general case for Y being any univariate Gaussian random variable.

From Expectations for absolutely continuous and discrete random variables by Rafi Muflih Abdur we state that $\mathbb{E}(e^Y) = \int_{\mathbb{R}} e^y \, \Pi(y) \, \mathrm{d}y$. For simplicity, let $\bar{y} := \mathbb{E}(Y), \sigma^2 := \mathrm{Var}(Y)$.

Evaluating the integral we obtain

$$\begin{aligned}
\mathbb{E}(e^Y) &= \int_{\mathbb{R}} e^y \Pi(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \exp(y) \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{1}{2\sigma^2}(y - \bar{y})^2\Big) \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{1}{2\sigma^2}(y - \bar{y})^2 + y\Big) \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{1}{2\sigma^2}(y - \bar{y})^2 - \frac{1}{2\sigma^2}(-2\sigma^2 y)\Big) \, \mathrm{d}y \\
&= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{1}{2\sigma^2}\big((y - \bar{y})^2 - 2\sigma^2 y\big)\Big) \, \mathrm{d}y \,.
\end{aligned}$$

Now we complete the square in the integral expression

$$\begin{aligned}
\mathbb{E}(e^Y) &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{1}{2\sigma^2}\big((y - (\bar{y} + \sigma^2))^2 - 2\bar{y}\sigma^2 - \sigma^4\big)\Big) \, \mathrm{d}y \\
&= \exp(\bar{y} + \frac{1}{2}\sigma^2) \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{1}{2\sigma^2}\big((y - (\bar{y} + \sigma^2))^2\big)\Big) \, \mathrm{d}y \,.
\end{aligned}$$

Notice that the expression in the integral is a PDF of a univariate Gaussian random variable $N(\bar{y} + \sigma^2, \sigma^2)$. Hence, the integral equals one.

$$\begin{aligned}
\mathbb{E}(e^Y) &= \exp(\bar{y} + \frac{1}{2}\sigma^2) \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{1}{2\sigma^2}\big((y - (\bar{y} + \sigma^2 y))^2\big)\Big) \, \mathrm{d}y \\
&= \exp(\bar{y} + \frac{1}{2}\sigma^2) \cdot 1 \\
&= \exp(\bar{y} + \frac{1}{2}\sigma^2).
\end{aligned}$$

Therefore, we had proved that for the univariate Gaussian random variable Y the MGF for the first moment has the form

$$\mathbb{E}(e^Y) = \exp(\bar{y} + \frac{1}{2}\sigma^2), \tag{2}$$

where $\bar{y} = \mathbb{E}(Y), \sigma^2 = \text{Var}(Y)$.

We shall now make two important observations about the mean and the variance of a linear combination of random variables. The mean of the linear combination is

$$\mathbb{E}(a \cdot X) = \mathbb{E}(a_1 X_1 + ... + a_N X_N) = a_1 \bar{x_1} + ... + a_N \bar{x_N} = a \cdot \bar{x} = a \cdot \mathbb{E}(X), \tag{3}$$

where $\bar{x}$ is the mean vector of X. Furthermore, since we denoted the linear combination $a \cdot X$ as a univariate Gaussian random variable Y, one infers that $a \cdot \bar{x} = \bar{y}$. Namely,

$$\bar{y} = a \cdot \mathbb{E}(X). \tag{4}$$

The variance is obtained using linearity of expectation

$$\begin{aligned}
\sigma^2 = \text{Var}(Y) = \text{Var}(a \cdot X) &= \mathbb{E}\Big((a \cdot (X - \bar{x}))^2\Big) \\
&= \mathbb{E}\Big((a^T(X - \bar{x}))(a^T(X - \bar{x}))^T\Big) \\
&= a^T \, \mathbb{E}\Big((X - \bar{x})(X - \bar{x})^T\Big) a \\
&= a^T \, \text{Cov}(X) \, a.
\end{aligned}$$

Namely,

$$\sigma^2 = a^T \, \text{Cov}(X) \, a. \tag{5}$$

Now all that is left is to rewrite the result (2) using (4) and (5).

$$\begin{aligned}
\mathbb{E}(e^{a \cdot X}) &= \mathbb{E}(e^Y) \\
&= \exp(\bar{y} + \frac{1}{2}\sigma^2) \\
&= \exp(a \cdot \mathbb{E}(X) + \frac{1}{2} a^T \, \text{Cov}(X) \, a) \quad \square
\end{aligned}$$

Observation: Following the result (3) $\mathbb{E}(aX) = a\mathbb{E}(X)$ (note that X is now a univariate Gaussian random variable and $a \in \mathbb{R}$) and using $\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$ one may notice that the change of variable in (2) $aX := Y$ for some $t \in \mathbb{R}$ yields the general form of the MGF of a univariate Gaussian random variable Y

$$\mathbb{E}(e^{tY}) = \exp(t \, \mathbb{E}(Y) + \frac{1}{2} t^2 \text{Var}(Y)). \tag{6}$$

Consider "if" direction: Let $a \in \mathbb{R}^N$, $t \in \mathbb{R}$. We know that

$$\mathbb{E}(e^{t(a \cdot X)}) = \exp(t \, \mathbb{E}(a \cdot X) + \frac{1}{2} t^2 a^T \text{Cov}(X) a).$$

Set $Y := a \cdot X$. Then we have

$$\mathbb{E}(e^{tY}) = \exp(t \, \mathbb{E}(Y) + \frac{1}{2} t^2 \text{Var}(Y)).$$

By Theorem 1.3.4 from [Ric], since $\mathbb{E}(e^{tY})$ exists $\forall t \in \mathbb{R}$, it defines Y uniquely. Since $\mathbb{E}(e^{tY})$ has the form of the MGF of a univariate Gaussian random variable (6), then by uniqueness, $Y = N\Big(\mathbb{E}(Y), \text{Var}(Y)\Big)$

$\implies a \cdot X = N\Big(\mathbb{E}(Y), \text{Var}(Y)\Big) \; \forall a \in \mathbb{R}$

$\implies X$ is a Gaussian vector. $\square$

References

[Arg] J.-L. Arguin, A first course in stochastic calculus.

[Ric] S. Richard, Introduction to Stochastic Calculus: Cumulative notes.