Al Kafi Muhtasin - 062301778

Exercise 3.2.2: Let $\{\mathcal{F}_t\}_{t\in\tau}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let X be a univariate random variable on this space. Set $X_t := \mathbb{E}(X|\mathcal{F}_t)$. Show that $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\tau}, (X_t)_{t\in\tau})$ is a martingale.

For this proof, we take the necessary assumption that $\mathbb{E}(|X|)<\infty$.

Now, $\mathbb{E}(|X_t|) = \mathbb{E}(|\mathbb{E}(X|\mathcal{F}_t)|) \leq \mathbb{E}(\mathbb{E}(|X||\mathcal{F}_t))$ [Prop. 3.1.3.(7)]

$= \mathbb{E}(|X|) < \infty$ [Eq. 3.1.6]

$\therefore X_t \in L^1(\Omega, \mathcal{F}, \mathbb{P})$

For any $s\leq t$, we have $X_s = \mathbb{E}(X|\mathcal{F}_s)$.

Also, $\mathcal{F}_s \subset \mathcal{F}_t$ according to definition of filtration.

$\therefore \mathbb{E}(\mathbb{E}(X|\mathcal{F}_t)|\mathcal{F}_s) = \mathbb{E}(X|\mathcal{F}_s)$ [Prop. 3.1.3.(5)]

$\Rightarrow \mathbb{E}(X_t|\mathcal{F}_s) = X_s \quad \forall s\leq t$

$\therefore (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\tau}, (X_t)_{t\in\tau})$ is a martingale.

Exercise 3.2.3: Consider $T=\mathbb{N}$ and a sequence $(X_n)_{n\in\mathbb{N}}$ of independent and real valued random variables satisfying $\mathbb{E}(X_n)=0$. Set $Y_n := \sum_{j=1}^{n} X_j$. Show that $(Y_j)_{j\in\mathbb{N}}$ and the natural filtration define a martingale

For this proof, we take the necessary assumption that $E(|X_n|)<\infty$ such that

$\mathbb{E}(|Y_n|)=\mathbb{E}(|X_1+X_2+\dots+X_n|)\leq\mathbb{E}(|X_1|+|X_2|+\dots+|X_n|)=\mathbb{E}(|X_1|)+\mathbb{E}(|X_2|)+\dots\mathbb{E}(|X_n|)<\infty.$

Then, $Y_n\in L^1(\Omega,\mathcal{F},\mathbb{P})$

Naturally, Y_n is adapted to $\mathcal{F}_n$, the natural filtration associated with Y_n.

Moreover, $\sigma(Y_n)$ contains the cumulative information upto $\sigma(X_n)$ as well as the information in $\sigma(Y_{n-1})$ by definition. Thus, the random vectors $(Y_1,Y_2,\dots,Y_n)$ and $(X_1,X_2,\dots,X_n)$ contain the same information and are adapted to same filtration

Now, $\forall m\leq n$, $\mathbb{E}(Y_n|\mathcal{F}_m)=\mathbb{E}(Y_m|\mathcal{F}_m)+\mathbb{E}(X_{m+1}|\mathcal{F}_m)+\dots+\mathbb{E}(X_n|\mathcal{F}_m)$

$=Y_m+\mathbb{E}(X_{m+1})+\dots+\mathbb{E}(X_n)$ [Prop. 3.1.3 (2) & (6)]

$=Y_m+0+\dots+0=Y_m$

$\therefore \mathbb{E}(Y_n|\mathcal{F}_m)=Y_m\ \forall m\leq n$

$\therefore (Y_n)_{n\in\mathbb{N}}$ with natural filtration defines a martingale.