How to make measure

Hideto Tsubouchi

In this report, we think about how to make a measurable space.

Overall Roadmap

1. Outer Measure
2. Measurable set
3. Carathéodory Theorem

1. Outer Measure

Def X : set, 2^X : Power set of X

Map $\nu : 2^X \to [0,\infty]$ is an outer measure.

$$\overset{\text{def}}{\iff} \begin{cases} \text{(i) } \nu(\phi) = 0 \\ \text{(ii) } E \subseteq F \ (\subseteq X) \Rightarrow \nu(E) \leq \nu(F) \\ \text{(iii) } \{E_j\}_{j=1}^{\infty} \subseteq 2^X \Rightarrow \nu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \nu(E_j) \end{cases}$$

☆ How to make an outer measure from a map

- We have $\rho : 2^X \to [0, \infty]$ satisfying $\rho(\phi) = 0$.
- We have $E \subseteq X$.

Then, if we define $\nu(E)$ by

$$\nu(E) := \inf \left\{ \sum_{k=1}^{\infty} \rho(A_k) \,\middle|\, \{A_k\}_{k=1}^{\infty} \subseteq 2^X,\ \bigcup_{k=1}^{\infty} A_k \supseteq E \right\}$$

$\nu : 2^X \to [0, \infty]$ is an outer measure.

☺

(i) $\phi \subseteq \bigcup_{j=1}^{\infty} \phi$, $\phi \subseteq X$

So, by the definition of ν,

$$0 \leq \nu(\phi) \leq \sum_{j=1}^{\infty} \rho(\phi) = 0 \quad (\because \rho(\phi) = 0)$$

$\therefore \ \nu(\phi) = 0$

(ii) We have two subsets of X, E and F satisfying $E \subseteq F$.

- $\forall \varepsilon > 0,\ \exists \{A_j\}_{j=1}^{\infty} \subseteq 2^X$;

$$F \subseteq \bigcup_{j=1}^{\infty} A_j \text{ and } \sum_{j=1}^{\infty} \rho(A_j) < \nu(F) + \varepsilon$$

- $E \subseteq F \subseteq \bigcup_{j=1}^{\infty} A_j \Rightarrow \nu(E) \leq \sum_{j=1}^{\infty} \rho(A_j)$

$\therefore \quad \nu(E) \leq \sum_{j=1}^{\infty} \rho(A_j) < \nu(F) + \varepsilon$

$\xrightarrow{\varepsilon \to 0} \nu(E) \leq \nu(F)$

(iii) $\{E_j\}_{j=1}^{\infty} \subseteq 2^X$

- $\forall \varepsilon > 0, \forall j \in \mathbb{N}, \exists \{A_{jk}\}_{k=1}^{\infty} \subseteq 2^X$;

  $E_j \subseteq \bigcup_{k=1}^{\infty} A_{jk}$ and $\sum_{k=1}^{\infty} \rho(A_{jk}) < \nu(E_j) + \frac{\varepsilon}{2^j}$

- $\bigcup_{j=1}^{\infty} E_j \subseteq \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} A_{jk}$

$$\rightarrow \nu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \rho(A_{jk})$$

$$< \sum_{j=1}^{\infty} \left(\nu(E_j) + \frac{\varepsilon}{2^j}\right)$$

$$= \sum_{j=1}^{\infty} \nu(E_j) + \varepsilon$$

$$\xrightarrow{\varepsilon \to 0} \nu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \nu(E_j) \qquad \boxtimes$$

In this way, if we are given a set X and map $\rho: 2^X \to [0, \infty]$ we can make an outer measure.

② Measurable Set

Def X: set ν: outer measure on X

$E \subseteq X$ is ν-measurable

$\overset{\text{def}}{\iff} \forall A \subseteq X,\ \nu(A) = \nu(A \cap E) + \nu(A - E)$

※ To prove $E \subseteq X$ is ν-measurable

We have only to prove "$\geq$" part.

$\because$

"$\leq$" is trivial by definition:

$A = (A \cap E) \cup (A - E)$ and ν is outer measure

$\Rightarrow \nu(A) \leq \nu(A \cap E) + \nu(A - E)$ ($\because$ (iii))

③ Carathéodory Theorem

Carathéodory Theorem shows us how to make measurable space.

Before seeing the theorem, let's review the concept of σ-algebra and measure.

Def X : set $\mathcal{S} \subseteq 2^X$

$\mathcal{S}$ is σ-algebra

$$\overset{\text{def}}{\Longleftrightarrow} \begin{cases} \cdot\ E \in \mathcal{S} \Rightarrow E^c \in \mathcal{S} \\ \cdot\ \{E_j\}_{j=1}^{\infty} \in \mathcal{S} \Rightarrow \bigcup_{j=1}^{\infty} E_j \in \mathcal{S} \end{cases}$$

Def $\mathcal{S}$: σ-algebra on X

$\mu : \mathcal{S} \to [0, \infty]$ is measure on $\mathcal{S}$

$$\Longleftrightarrow \begin{cases} \cdot\ \mu(\phi) = 0 \\ \cdot\ \{E_j\}_{j=1}^{\infty} \subseteq \mathcal{S} : \text{disjoint} \Rightarrow \mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j) \end{cases}$$

Thm (Carathéodory)

If we have ν (outer measure on X) and $\mathcal{M}$ (collection of ν-measurable sets) then,

(i) $\mathcal{M}$ is σ-algebra

(ii) $$\begin{array}{ccccc} \mu : & \mathcal{M} & \longrightarrow & [0, \infty] & \text{is measure on } \mathcal{M}. \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ & E & \longmapsto & \nu(E) & \end{array}$$

proof

(i) $\mathcal{M}$ is σ-algebra

(1) $E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$

$\because$ $\forall A \subseteq X$, $\nu(A) = \nu(A \cap E) + \nu(A - E)$ $(\because E \in \mathcal{M})$

$$= \nu(A - E^c) + \nu(A \cap E^c)$$

$\therefore$ E^c is ν-measurable $\rightarrow E^c \in \mathcal{M}$

(2) $E, F \in \mathcal{M} \Rightarrow E \cup F \in \mathcal{M}$

$\because$ $\forall A \subseteq X$,

$$\nu(A) = \nu(A \cap E) + \nu(A - E) \quad (\because E \in \mathcal{M})$$

$$= \big(\nu((A \cap E) \cap F) + \nu((A \cap E) - F)\big)$$

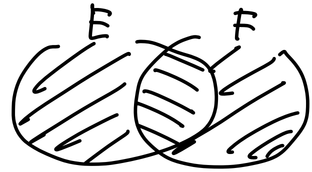
$$+ \big(\nu((A - E) \cap F) + \nu((A - E) - F)\big) \quad (\because F \in \mathcal{M})$$

$$= \nu(A \cap (E \cap F)) + \nu(A \cap (E \cap F^c))$$

$$+ \nu(A \cap (E^c \cap F)) + \nu(A \cap (E^c \cap F^c)) \quad \cdots ①$$

$$(E\cap F)\cup(E\cap F^c)\cup(E^c\cap F)=E\cup F$$

$$\Downarrow$$



$$A\cap(E\cup F)=A\cap\{(E\cap F)\cup(E\cap F^c)\cup(E^c\cap F)\}$$
$$=\{A\cap(E\cap F)\}\cup\{A\cap(E\cap F^c)\}\cup\{A\cap(E^c\cap F)\}$$

So, by definition of ν,

$$\nu(A\cap(E\cup F))\le\nu(A\cap(E\cap F))+\nu(A\cap(E\cap F^c))+\nu(A\cap(E^c\cap F)) \quad \cdots ③$$

By ①, ②, we can get below

$$\nu(A)=\{\nu(A\cap(E\cap F))+\nu(A\cap(E\cap F^c))+\nu(A\cap(E^c\cap F))\}+\nu(A\cap(E\cup F)^c)$$
$$\ge\nu(A\cap(E\cup F))+\nu(A-(E\cup F))$$

By the definition

$\therefore$ $E\cup F$ is ν-measurable $\to$ $E\cup F\in\mathcal{M}$.

(3) $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$: disjoint $\quad E := \bigcup_{j=1}^{\infty} E_j$

Then,

$$\nu(A) \geq \sum_{j=1}^{\infty} \nu(A \cap E_j) + \nu(A - E) \quad \text{for all } A \subseteq X$$

$(\because)$

If we define G_n for all $n \in \mathbb{N}$

by $G_n := \bigcup_{j=1}^{n} E_j$, $\quad G_n \in \mathcal{M} \quad (\because (2))$

By $E_n \in \mathcal{M}$,

$$\begin{aligned}
\nu(A \cap G_n) &= \nu(\underbrace{(A \cap G_n) \cap E_n}_{\substack{= A \cap (G_n \cap E_n) \\ = A \cap E_n}}) + \nu(\underbrace{(A \cap G_n) - E_n}_{\substack{= A \cap (G_n - E_n) \\ = A \cap G_{n-1}}}) \\
&= \nu(A \cap E_n) + \nu(A \cap G_{n-1})
\end{aligned}$$

As above,

$$\nu(A \cap G_{n-1}) = \nu(A \cap E_{n-1}) + \nu(A \cap G_{n-2})$$

$$\vdots$$

If we have the same operation n times,

we have $\nu(A \cap G_n) = \sum_{j=1}^{n} \nu(A \cap E_j) \quad (\because G_1 = E_1)$

On the other hand

$$\nu(A) = \nu(A \cap G_n) + \nu(A - G_n) \quad (\because G_n \in \mathcal{M})$$

$$G_n = \bigcup_{j=1}^{n} E_j \subseteq \bigcup_{j=1}^{\infty} E_j = E$$

$$\rightarrow A - G_n \supseteq A - E$$

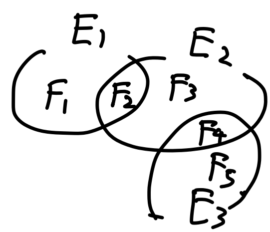
$$\rightarrow \nu(A - G_n) \geq \nu(A - E)$$

$$\therefore \nu(A) = \nu(A \cap G_n) + \nu(A - G_n)$$
$$\geq \sum_{j=1}^{n} \nu(A \cap E_j) + \nu(A - E)$$

$$\xrightarrow{n \to \infty} \nu(A) \geq \sum_{j=1}^{\infty} \nu(A \cap E_j) + \nu(A - E)$$

(4) $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M} \Rightarrow E := \bigcup_{j=1}^{\infty} E_j \in \mathcal{M}$

$\because$



If we define $\{F_n\}_{n=1}^{\infty}$ by

$$F_n := \begin{cases} E_1 & (n=1) \\ E_n - \bigcup_{j=1}^{n-1} E_j & (n \geq 2) \end{cases}$$

$\{F_n\}_{n=1}^{\infty}$ is disjoint and $\bigcup_{n=1}^{\infty} F_n = E$

So, by (3),

$\forall A \subseteq X, \quad \nu(A) \geq \sum_{j=1}^{\infty} \nu(A \cap F_j) + \nu(A - E)$

On the other hand,

$$A \cap E = A \cap \left(\bigcup_{j=1}^{\infty} F_j \right) = \bigcup_{j=1}^{\infty} (A \cap F_j)$$

$\rightarrow \nu(A \cap E) \leq \sum_{j=1}^{\infty} \nu(A \cap F_j) \quad (\because \text{(iii)})$

$$\therefore \nu(A) \geq \sum_{j=1}^{\infty} \nu(A \cap F_j) + \nu(A - E)$$
$$\geq \nu(A \cap E) + \nu(A - E)$$

$\rightarrow E = \bigcup_{j=1}^{\infty} E_j \in \mathcal{M}$

(ii) Map μ defined by $\mu : \mathcal{M} \rightarrow [0, \infty]$, $E \mapsto \nu(E)$

is measure on $\mathcal{M}$.

$\because$

- $\mu(\phi) = \nu(\phi) = 0 \quad (\because \text{(i)})$
- $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$: disjoint $\Rightarrow \mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \mu(E_j)$

$\because$

$E := \bigcup_{j=1}^{\infty} E_j$

- $$\nu(E) \geq \sum_{j=1}^{\infty} \nu(E \cap E_j) + \underbrace{\nu(E - E)}_{= 0}$$
  $$= \sum_{j=1}^{\infty} \nu(E_j)$$

- $$\nu(E) = \nu\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} \nu(E_j) \quad (\because \text{(iii)})$$

$$\therefore \quad \mu(E) = \nu(E) = \sum_{j=1}^{\infty} \nu(E_j) = \sum_{j=1}^{\infty} \mu(E_j)$$