On Feynman-Kac formula with terminal value

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Special Mathematics Lecture: Introduction to Stochastic Calculus (Fall 2023)

Langevin-Type SDE ([2], pg. 132)

Let X_t satisfy the stochastic differential equation (SDE)

$$\mathrm{d}X_t = a_t X_t \mathrm{d}t + \mathrm{d}B_t, \quad (5.32) \tag{1}$$

where a_t is a given adapted and continuous process. When $a_t = -\alpha$, the equation is the Langevin equation.

We solve the SDE in two ways: by using the general solution ([3], **Eq. 5.2.7**, pg. 47), and directly, similarly to Langevin's SDE.

1. Clearly, $\beta_t = a_t$, $\gamma_t = 1$, and $\alpha_t = \delta_t = 0$. In order to find U_t we must solve $\mathrm{d}U_t = a_t U_t \mathrm{d}t$, which gives $U_t = e^{\int_0^t a_s \mathrm{d}s}$. Thus from ([3], **Eq. 5.2.7**, pg. 47)

$$X_t = e^{\int_0^t a_s \mathrm{d}s}\left(X_0 + \int_0^t e^{-\int_0^u a_s \mathrm{d}s}\mathrm{d}B_u\right).$$

2. Consider the process $e^{-\int_0^t a_s \mathrm{d}s}X_t$ and use integration by parts. The process $e^{-\int_0^t a_s \mathrm{d}s}$ is continuous and is of finite variation. Therefore, it has zero covariation with X_t, hence

$$\mathrm{d}\left(e^{-\int_0^t a_s \mathrm{d}s}X_t\right) = e^{-\int_0^t a_s \mathrm{d}s}\mathrm{d}X_t - a_t e^{-\int_0^t a_s \mathrm{d}s}X_t \mathrm{d}t = e^{-\int_0^t a_s \mathrm{d}s}\mathrm{d}B_t.$$

   Integrating, we obtain

$$e^{-\int_0^t a_s \mathrm{d}s}X_t = X_0 + \int_0^t e^{-\int_0^u a_s \mathrm{d}s}\mathrm{d}B_u,$$

   and finally,

$$X_t = X_0 e^{\int_0^t a_s \mathrm{d}s} + e^{\int_0^t a_s \mathrm{d}s}\left(\int_0^t e^{-\int_0^u a_s \mathrm{d}s}\mathrm{d}B_u\right).$$

Theorem 1 (Feynman-Kac formula with terminal value ([3], pg. 57))**.** *Consider an Itô process satisfying the differential stochastic equation*

$$\mathrm{d}X_t = \sigma(t, X_t)\,\mathrm{d}B_t + \mu(t, X_t)\,\mathrm{d}t, \tag{2}$$

and let $r : [0,T] \times \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *be measurable and bounded functions. Assume that* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *is a solution of the equation*

$$[\partial_t f](t,x) + [L_t f](t,x) = r(t,x)f(t,x), \quad f(T,y) = g(y). \tag{3}$$

Then, f *is unique and satisfies the relation*

$$f(t,y) = \mathbb{E}\left[\exp\left(-\int_t^T r(u,X_u)\,\mathrm{d}u\right) g(X_T) \,\middle|\, X_t = y\right].$$

Proof. Before proving the theorem, consider the differential form of Itô's formula for the given Itô process (2) ([3], **Proposition 5.1.4** pg. 43)

$$\mathrm{d}f(t,X_t) = \left([\partial_t f](t,X_t) + \frac{1}{2}\sigma^2(t,X_t)[\partial_x^2 f](t,X_t)\right)\mathrm{d}t + [\partial_x f](t,X_t)\,\mathrm{d}X_t.$$

Then we define a second-order differential equation for the given Itô process (2), $L_t := \frac{1}{2}\sigma^2(t,x)\partial_x^2 + \mu(t,x)\partial_x$. Therefore, one can rewrite the equation above into

$$\begin{aligned}\mathrm{d}f(t,X_t) &= \left([\partial_t f](t,X_t) + \frac{1}{2}\sigma^2(t,X_t)[\partial_x^2 f](t,X_t)\right)\mathrm{d}t + [\partial_x f](t,X_t)(\sigma(t,X_t)\,\mathrm{d}B_t + \mu(t,X_t)\,\mathrm{d}t)\\ &= \Big([\partial_t f](t,X_t) + [L_t f](t,X_t)\Big)\mathrm{d}t + \sigma(t,X_t)[\partial_x f](t,X_t)\,\mathrm{d}B_t.\end{aligned}$$

We give a sketch of the proof by using Itô's formula coupled with solutions of a linear stochastic differential equation (SDE). Take a solution to (3) and apply Itô's formula:

$$\mathrm{d}f(t,X_t) = \Big([\partial_t f](t,X_t) + [L_t f](t,X_t)\Big)\mathrm{d}t + [\partial_x f](t,X_t)\sigma(t,X_t)\,\mathrm{d}B_t.$$

The last term is a martingale term, so write it as $\mathrm{d}M_t$. Now use (3) to obtain

$$\mathrm{d}f(t,X_t) = r(t,X_t)f(t,X_t)\,\mathrm{d}t + \mathrm{d}M_t.$$

This is a linear SDE of Langevin type for $f(t,X_t)$ where B_t is replaced by M_t. Integrating this SDE between t and T, we obtain

$$f(T,X_T) = f(t,X_t)e^{\int_t^T r(u,X_u)\,\mathrm{d}u} + e^{\int_t^T r(u,X_u)\,\mathrm{d}u}\int_t^T e^{\int_t^s r(u,X_u)\,\mathrm{d}u}\,\mathrm{d}M_s.$$

However, we know that $f(T,X_T) = g(X_T)$, and rearranging the equation above, we obtain

$$g(X_T)e^{-\int_t^T r(u,X_u)\,\mathrm{d}u} = f(t,X_t) + \int_t^T e^{\int_t^s r(u,X_u)\,\mathrm{d}u}\,\mathrm{d}M_s.$$

As mentioned in ([3], **Theorem 4.3.1**, pg. 37), the last term on the r.h.s. is an integral of a bounded function with respect to the martingale which defines a mean zero and continuous martingale map[1]. Taking the expectation given $X_t = y$, we obtain $f(t,y)$. □

[1] For the proof that $\sigma(t,X_t)[\partial_x f](t,X_t)$ belongs to $M^2([0,T])$, refer to [2], **Theorem 6.2**, pg. 152. The main idea of the proof is to utilize the fact that $\partial_x f(t,x)$ is bounded for all t and x.

Exercise 6.3.6 ([3], pg. 57)

Give a probabilistic representation of the solution of the equation

$$\partial_t f + \frac{1}{2}\sigma^2 x^2 \partial_x^2 f + \mu x \partial_x f = rf, \quad f(T,y) = y^2,$$

for $r, \sigma, \mu > 0$*.*

In this equation, $L_t = \frac{1}{2}\sigma^2 x^2 \partial_x^2 f + \mu x \partial_x f$, which is the infinitesimal generator of the geometric Brownian motion given by the SDE

$$\mathrm{d}X_t = \sigma X_t \,\mathrm{d}B_t + \mu X_t \,\mathrm{d}t.$$

The solution to this equation is ([1], **pg. 1**)

$$X_t = X_0 e^{(\mu - \sigma^2/2)t + \sigma B_t}.$$

By the Feynman-Kac formula

$$f(t,y) = \mathbb{E}\left[e^{-r(T-t)} X_T^2 \,\middle|\, X_t = y\right] = e^{-r(T-t)} \mathbb{E}\left[X_T^2 \middle| \, X_t = y\right].$$

Using $X_T = X_t e^{(\mu - \sigma^2/2)(T-t) + \sigma(B_T - B_t)}$, we obtain

$$\begin{aligned} f(t,y) &= e^{-r(T-t)} \mathbb{E}\left[X_T^2 \middle| \, X_t = y\right] \\ &= e^{-r(T-t)} y^2 e^{2[(\mu - \sigma^2/2)(T-t) + \sigma(B_T - B_t)]} \\ &= y^2 e^{(2\mu + \sigma^2 - r)(T-t)}. \end{aligned}$$

Exercise 6.10 ([2], pg. 184)

Give a probabilistic representation of the solution $f(x,t)$ of the Partial Differential Equation (PDE)

$$\frac{1}{2}\partial_x^2 f + \partial_t f = 0, \quad 0 \le t \le T, \quad f(x,T) = x^2.$$

Solve this PDE using the solution of the corresponding stochastic differential equation.

To solve the given exercise, we first identify the SDE that corresponds to the PDE. For a PDE of the form

$$\frac{1}{2}\partial_x^2 f + \partial_t f = 0,$$

the corresponding SDE for a standard Brownian motion is

$$\mathrm{d}X_t = \mathrm{d}B_t,$$

Applying the Feynman-Kac formula, we represent the solution to the PDE as the expected value of the terminal condition, leading to

$$f(t,x) = \mathbb{E}\left[X_T^2 \,\middle|\, X_t = x\right].$$

Given that $X_T - X_t = B_T - B_t \implies X_T = x + (B_T - B_t)$ and using the properties of Brownian motion, we compute

$$f(t,x) = \mathbb{E}\left[(x + B_T - B_t)^2\right] = x^2 + \mathbb{E}[(B_T - B_t)^2].$$

Since $B_T - B_t$ is normally distributed with mean 0 and variance $T - t$, we find

$$f(t,x) = x^2 + (T - t).$$

Thus, the probabilistic representation of the solution to the PDE is

$$f(t,x) = x^2 + (T - t),$$

which concludes the solution to this exercise.

References

[1] Rafi R Firdaus and Ngo Gia Linh. On black-scholes equation. link.

[2] Fima C Klebaner. *Introduction to stochastic calculus with applications*. World Scientific Publishing Company, 2012.

[3] Serge Richard. *Special Mathematics Lecture: Introduction to Stochastic Calculus*. 2023.