On Black-Scholes Equation

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Exercise 5.2.2.

Let X be an Itô process satisfying

$$\mathrm{d}X_t = rX_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t, \quad X(0) = 1$$

Take $f(t,x) = \ln x$, then $[\partial_t f](t,x) = 0$, $[\partial_x f](t,x) = \frac{1}{x}$, $[\partial_x^2 f](t,x) = -\frac{1}{x^2}$, $f(0,X_0) = \ln 1 = 0$. By Proposition 5.1.4 (Itô's lemma for Itô process)

$$f(t,X_t) = f(0,X_0) + \int_0^t \frac{1}{X_u}\,\mathrm{d}X_u + \int_0^t \left[0 + \frac{1}{2}\sigma^2 X_u^2 \left(\frac{-1}{X_u^2}\right)\right]\mathrm{d}u$$

From which one gets

$$\begin{aligned}
\ln X_t &= 0 + \int_0^t \frac{1}{X_u}\,\mathrm{d}X_u - \int_0^t \frac{1}{2}\sigma^2\,\mathrm{d}u \\
&= \int_0^t \frac{1}{X_u}\,(rX_u\,\mathrm{d}u + \sigma X_u\,\mathrm{d}B_u) - \int_0^t \frac{1}{2}\sigma^2\,\mathrm{d}u \\
&= \int_0^t \left(r - \frac{1}{2}\sigma^2\right)\mathrm{d}u + \int_0^t \sigma\,\mathrm{d}B_t \\
&= \left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t.
\end{aligned}$$

The solution for this equation is

$$X_t = e^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t}.$$

Then, for X an Itô process, we want to consider the following equation

$$\mathrm{d}X_t = r_t X_t\,\mathrm{d}t, \quad X(0) = 1$$

The solution for this equation is the same thing as what we did previously, with $\sigma = 0$

$$\begin{aligned}
\ln X_t &= \int_0^t r_s\,\mathrm{d}s \\
\implies X_t &= \exp\left(\int_0^t r_s\,\mathrm{d}s\right).
\end{aligned}$$

Black-Scholes model

One of the simplest examples of a market model is given for $N = 1$ and $n = 1$ by the stochastic differential equation

$$\mathrm{d}S_t^1 = \mu S_t^1 \,\mathrm{d}t + \sigma S_t^1 \,\mathrm{d}B_t \tag{1}$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$. The constant μ is called the mean rate of return, and the constant σ is the volatility.

On the other hand, the stochastic process for S^0 is given by

$$\mathrm{d}S_t^0 = r_t S_t^0 \,\mathrm{d}t \tag{2}$$

Note that the function r can also be deterministic, which is independent of the underlying probability space associated with the Brownian motion. The first equation corresponds to time-homogeneous diffusion processes, as introduced in Definition 5.1.2, while the second equation is a time-inhomogeneous diffusion process. The solutions of these equations are known: the first one has already been mentioned in Exercise 5.2.2 above and reads

$$S_t^1 = S_0^1 e^{(\mu - \sigma^2/2)t + \sigma B_t} \tag{3}$$

with S_0^1 an initial condition independent of the Brownian motion, while

$$S_t^0 = S_0^0 \exp\left(\int_0^t r_s \,\mathrm{d}s\right). \tag{4}$$

In the special case of a constant parameter r, then the solution is simply $S_t^0 = S_0^0 e^{rt}$ for $t \in [0, T]$.