REPORT 3. ON HOMOGENEOUS MARKOV PROPERTY.

NGO GIA LINH (0420014463)

Exercise 6.1.4 Show that Brownian motion has the time-homogeneous Markov property.

First, recall the definition of 1-D Brownian motion

A stochastic process $B := (\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t, B_t)$ taking values in $\mathbb{R}$ is a 1-D Brownian motion if

1. $B_0 = 0$ a.s
2. For any $0 \leq s < t$ the random variable $B_t - B_s$ is independent of $\mathcal{F}_s$
3. For any $0 \leq s < t$ the random variable $B_t - B_s$ is a Gaussian random variable $N(0, t-s)$

Consider $g(B_t)$ for some time t and bounded function g. and a random variable W that is $\mathcal{F}_s$ - measurable for $s < t$.

We have:

$$E[g(B_t)W] = E(g(B_s + \underbrace{B_t - B_s}_{Y})W]$$

Since $B_t - B_s \sim N(0, t-s)$ and is independent of $\mathcal{F}_s$ (as stated in 2 and 3)

$$E[g(B_s+Y)W] = E\left[\int_{\mathbb{R}} g(B_s+y)W \cdot \frac{e^{-\frac{y^2}{2(t-s)}}}{\sqrt{2(t-s)}}\, dy\right]$$

Fubini's Theorem $\leftarrow$ $$= \int_{\mathbb{R}} E[g(B_s+y)W] \frac{e^{-\frac{y^2}{2(t-s)}}}{\sqrt{2(t-s)}}\, dy.$$

Thus, the conditional expectation of $g(B_t)$ given $\mathcal{F}_s$ is

$$E[g(B_t) \mid \mathcal{F}_s] = E[g(B_s+Y) \mid \mathcal{F}_s]$$

$$= E_Y[g(B_s+Y)] \quad \text{since } Y \text{ is independent of } \mathcal{F}_s$$

$$= \int_{\mathbb{R}} g(B_s+y) \frac{e^{-\frac{y^2}{2(t-s)}}}{\sqrt{2(t-s)}}\, dy$$

R.H.S is a function of $t-s$ and B_s only, it does not depend on $\mathcal{F}_s$.
It implies that $E[g(B_t) \mid \mathcal{F}_s] = E[g(B_t) \mid B_s]$.

Since it depends on $t-s$ one has

$$E[g(B_t) \mid \mathcal{F}_s] = E[g(B_t) \mid B_s] = E[g(B_{t-s}) \mid B_0].$$

This can also be written as (by taking $g = \mathbb{1}_A$)

$$\boxed{\mathbb{P}(B_t \in A \mid \mathcal{F}_s) = \mathbb{P}(B_{t-s} \in A \mid B_0)}$$

Hence the standard Brownian motion has the time homogeneous Markov property. (see def. 6.1.2).