REPORT 2. SOME PROBLEMS ON ITO'S INTEGRAL

NGO HA LINH (042001463)

The following exercises are applications of proposition 4.4.4.

Proposition 4.4.4. *Let $B := (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (B_t)_{t\geq 0})$ be the standard 1-dimensional Brownian motion, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and with $\partial_t f$, $\partial_x f$, and $\partial_x^2 f$ (second derivative with respect to its space variable) also continuous. Then the following equality holds:*

$$f(t, B_t) = f(0, \mathbf{0}) + \int_0^t [\partial_x f](s, B_s)\,\mathrm{d}B_s + \int_0^t \left\{[\partial_t f](s, B_s) + \tfrac{1}{2}[\partial_x^2 f](s, B_s)\right\}\mathrm{d}s, \tag{4.4.2}$$

Exercise 4.4.6. *By considering the function $f(t, x) = tx^2$, write an expression (as simple as possible) for the Itô integral $\int_0^t sB_s\,\mathrm{d}B_s$, see also [12, Example 7.3.2].*

We have the derivatives as follows:

$\partial_t f = x^2$

$\partial_x f = 2tx$

$\partial_x^2 f = 2t$

By proposition 4.4.4.

$$f(t, B_t) = f(0,0) + \int_0^t [\partial_x f](s, B_s)\, dB_s + \int_0^t \{[\partial_t f](s, B_s) + \frac{1}{2}[\partial_x^2 f](s, B_s)\} ds$$

$$t \cdot B_t^2 = 0 + \int_0^t 2sB_s\, dB_s + \int_0^t (B_s^2 + \frac{1}{2} 2s)\, ds$$

$$= 2\int_0^t sB_s\, dB_s + \int_0^t B_s^2\, ds + \left.\frac{s^2}{2}\right|_0^t$$

$$\Leftrightarrow \int_0^t sB_s\, dB_s = \left(tB_t^2 - \int_0^t B_s^2\, ds - \frac{t^2}{2}\right)\frac{1}{2}$$

$$\Leftrightarrow \boxed{\int_0^t sB_s\, dB_s = \frac{1}{2} tB_t^2 - \frac{t^2}{4} - \frac{1}{2}\int_0^t B_s^2\, ds}$$

Exercise 4.4.7 (The Itô exponential). *Consider the function* $f(t,x) = e^{x-\frac{1}{2}t}$ *and show that*

$$\int_0^t f(s,B_s)\,dB_s = f(t,B_t) - f(0,0).$$

Observe that for the usual exponential function one has $\int_0^t e^s\,ds = e^t - e^0$, *and for this reason the function* f *is sometimes called the* Itô exponential.

We have the derivatives as follows:

$$\partial_x f(t,x) = e^{x-1/2t} = f(t,x)$$

$$\partial_t f(t,x) = \left(e^{x-\frac{1}{2}t}\right)\left(-\frac{1}{2}\right)$$

$$\partial_x^2 f(t,x) = e^{x-\frac{1}{2}t} = f(t,x)$$

By remark 4.4.5:

$$\int_0^t [\partial_x f](s,B_s)\,dB_s = f(t,B_t) - f(0,0) - \int_0^t \{[\partial_t f](s,B_s) + \frac{1}{2}[\partial_x^2 f](s,B_s)\}\,ds$$

$$\Leftrightarrow \int_0^t f(s,B_s)\,dB_s = f(t,B_t) - f(0,0) - \int_0^t \{-\frac{1}{2}e^{s-\frac{1}{2}B_s} + \frac{1}{2}e^{s-\frac{1}{2}B_s}\}\,ds$$
$$= f(t,B_t) - f(0,0) - 0$$
$$= f(t,B_t) - f(0,0)$$

$$\boxed{\int_0^t f(s,B_s)\,dB_s = f(t,B_t) - f(0,0)}$$

Now consider the function $f(t,x) = e^t$

$$\partial_x f(t,x) = 0$$
$$\partial_t f(t,x) = e^t$$
$$\partial_x^2 f(t,x) = 0$$

$$\int_0^t [\partial_x f](s,B_s)\,dB_s = f(t,B_t) - f(0,0) - \int_0^t \{[\partial_t f](s,B_s) + \frac{1}{2}[\partial_x^2 f](s,B_s)\}\,ds.$$
$$0 = e^t - e^0 - \int_0^t \{e^s + 0\}\,ds$$

$$\boxed{\int_0^t e^s\,ds = e^t - e^0}$$

Exercise 4.4.8. *Set* $f(t,x) := e^{(c-\frac{1}{2}\sigma^2)t+\sigma x}$ *for* $c \in \mathbb{R}$ *and* $\sigma > 0$, *and consider the process* $X_t := f(t, B_t)$. *Show that*

$$X_t = X_0 + c\int_0^t X_s \, ds + \sigma \int_0^t X_s \, dB_s.$$

We have the derivatives:

$$\partial_x f = e^{(c-1/2\sigma^2)t+\sigma x}\,\sigma = \sigma f(t,x)$$

$$\partial_t f = e^{(c-\frac{1}{2}\sigma^2)t+\sigma x}\,(c-\tfrac{1}{2}\sigma^2) = (c-\tfrac{1}{2}\sigma^2) f(t,x)$$

$$\partial_x^2 f = \sigma^2 e^{(c-\frac{1}{2}\sigma^2)} = \sigma^2 f(t,x)$$

$$X_t = X_0 + \int_0^t \underbrace{e^{(c-1/2\sigma^2)s+\sigma B_s}}_{f(s,B_s)=X_s}\cdot\sigma\, dB_s + \int_0^t \Big\{(c-\tfrac{1}{2}\sigma^2)\underbrace{e^{(c-\frac{1}{2}\sigma^2)s+\sigma B_s}}_{X_s} + \tfrac{1}{2}\underbrace{e^{(c-\frac{1}{2}\sigma^2)s+\sigma B_s}}_{X_s}\Big\}\sigma^2\, ds$$

$$= X_0 + \int_0^t \sigma X_s\, dB_s + \int_0^t \big[(c-\tfrac{1}{2}\sigma^2) X_s + \tfrac{1}{2}\sigma^2 X_s\big]\, ds$$

$$= X_0 + \sigma\int_0^t X_s\, dB_s + c\int_0^t X_s\, ds.$$

So $$\boxed{X_t = X_0 + c\int_0^t X_s\, ds + \sigma\int_0^t X_s\, dB_s}$$