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In this report, we will show the following statement

The random variables defined by $Z_t := tB_{1/t}$ for $t > 0$ and $Z_0 = 0$ define a natural Brownian motion.

First, let us define Brownian motion

Definition 2.4.1 (1-dimensional Brownian motion). *A Stochastic process $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ taking values in $\mathbb{R}$ is a* 1-dimensional Brownian motion *if*

1. $B_0 = \mathbf{0}$ *a.s.,*
2. *For any $0 \le s \le t$ the random variable $B_t - B_s$ is independent of $\mathcal{F}_s$,*
3. *For any $0 \le s < t$ the random variable $B_t - B_s$ is a Gaussian random variable $N(0, t-s)$.*

and its properties.

Proposition 2.4.4. *Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ be a 1-dimensional Brownian motion. Then*

1. $B_0 = \mathbf{0}$ *a.s.,*
2. *For every $0 \le t_1 < t_2 < \cdots < t_N$, the N-dimensional vector $B := (B_{t_1}, B_{t_2}, \ldots, B_{t_N})^T$ is a Gaussian vector with $\mathbb{E}(B) = 0$,*
3. $\mathbb{E}(B_t B_s) = t \wedge s$.

Now, let us show that $Z_t := tB_{1/t}$ possesses such properties in the definition (2.4.1)

1. $Z_0 = 0$ a.s given.

2. For any $0 \le s \le t$, the random variable $Z_t - Z_s$ is independent of $\mathcal{F}_s$.

To prove this statement, we evoke the following lemma:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $(X_t)_{t\ge 0}$ be a Gaussian process on $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that $E(X_t) = 0$ $\forall t \ge 0$. Then, the following statements are equivalent

1. $X_t - X_s$ is independent of $\sigma(X_r \mid r \leq s)$ for any $t \geq s \geq 0$

2. $\mathbb{E}((X_t - X_s)X_r) = 0$ for any $t \geq s \geq r \geq 0$.

The proof of this lemma is available on "On Independence" by Ziyu Liu.

To be in this framework, we need $(Z_t)_{t \in J}$ to be a Gaussian process.

We can write, for finite family $\{t_1, t_2, \cdots, t_N\} \subset J$,

$$\vec{Z}_N = (Z_{t_1}, Z_{t_2}, \cdots, Z_{t_N})^T$$
$$= (t_1 B_{1/t_1}, t_2 B_{1/t_2}, \cdots, t_N B_{1/t_N})^T$$

We know $\vec{Z}_N$ is a Gaussian vector since $\vec{B}_N = (B_{1/t_1}, B_{1/t_2}, \cdots, B_{1/t_N})^T$ is also a Gaussian vector. Hence, $(Z_t)_{t \in J}$ a Gaussian process.

Additionally $\mathbb{E}(Z_t) = \mathbb{E}(t B_{1/t}) = t\,\mathbb{E}(B_{1/t}) = 0$

Thus, Z_t fits into this framework

By showing that $\mathbb{E}((Z_t - Z_s)Z_r) = 0$ for any $t \geq s \geq r > 0$, we can say that $Z_t - Z_s$ is independent of $\mathcal{F}_s$.

For arbitrary $t \geq s \geq r > 0$:

$$\begin{aligned}
\mathbb{E}((Z_t - Z_s)Z_r) &= \mathbb{E}(Z_t Z_r - Z_s Z_r) \\
&= \mathbb{E}(Z_t Z_r) - \mathbb{E}(Z_s Z_r) \\
&= \mathbb{E}(t \cdot r \cdot B_{1/t} B_{1/r}) - \mathbb{E}(s \cdot r \; B_{1/s} B_{1/r}) \\
&= tr\,\mathbb{E}(B_{1/t} B_{1/r}) - sr\,\mathbb{E}(B_{1/s} B_{1/r}) \\
&= tr\,(1/t \wedge 1/r) - sr\,(1/s \wedge 1/r) \\
&= r - r \\
&= 0
\end{aligned}$$

For $t = s = r = 0$:

$$\mathbb{E}((Z_t - Z_s)Z_r) = \mathbb{E}(Z_t Z_r - Z_s Z_r)$$
$$= \mathbb{E}(Z_t Z_r) - \mathbb{E}(Z_s Z_r)$$
$$= 0$$

Hence $Z_t - Z_s$ independent of $\mathcal{F}_s$.

3. For any $0 \leq s \leq t$, the random variable $Z_t - Z_s$ is a Gaussian random variable $N(0, t-s)$

For X a Gaussian random variable, $X \sim N(\mu, \sigma^2)$.
i.e $X \sim N(\mathbb{E}(X), \mathbb{E}(X^2) - \mathbb{E}(X)^2)$

Now, replace X by $Z_t - Z_s$, we obtain
$N(\mathbb{E}(Z_t - Z_s), \mathbb{E}((Z_t - Z_s)^2) - (\mathbb{E}(Z_t - Z_s))^2)$

$$\mathbb{E}(Z_t - Z_s) = \mathbb{E}(tB_{1/t}) - \mathbb{E}(sB_{1/s})$$
$$= t\mathbb{E}(B_{1/t}) - s\mathbb{E}(B_{1/s})$$
$$= 0$$
$$\Rightarrow (\mathbb{E}(Z_t - Z_s))^2 = 0$$

$$\mathbb{E}((Z_t - Z_s)^2) = \mathbb{E}(Z_t^2) - 2\mathbb{E}(Z_t Z_s) + \mathbb{E}(Z_s^2)$$
$$= \mathbb{E}(t^2 B_{1/t} B_{1/t}) - 2\mathbb{E}(ts B_{1/t} B_{1/s}) + \mathbb{E}(s^2 B_{1/s} B_{1/s})$$
$$= t^2 \cdot \tfrac{1}{t} - 2ts\tfrac{1}{t} + s^2 \cdot \tfrac{1}{s}$$
$$= t - s$$

Finally, we obtain $Z_t - Z_s \sim N(0, t-s)$

Thus, we see that $Z_t : tB_{1/t}$ is indeed a Brownian motion.