Special Mathematics Lecture: Stochastic Calculus

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Exercise 1.1.2. *Prove this statement: if $\mathcal{F}$ is a collection of subsets of Ω with $\Omega \in \mathcal{F}$ and which is closed under complement and countable unions, then it is closed under countable intersections.*

For $\mathcal{F}$ closed under complement and countable unions, it has the following properties.

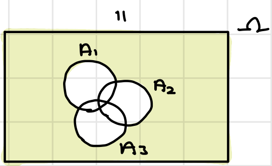
Let $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{F}$ and A_j^c be the compement of A_j with $A_j^c := \Omega \setminus A_j$.

Now,

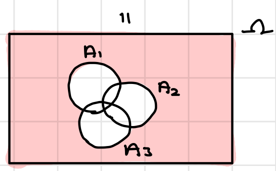
- Closed under compement means that for any $A_j \in \mathcal{F}$, $A_j^c \in \mathcal{F}$
- Closed under countable union means that for any union of A_j, eg. $B = (A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k)$ for some $k \in \mathbb{N}$, $B \in \mathcal{F}$

Let us express two combination of sets with diagrams.

1. $A_1^c \cap A_2^c \cap A_3^c$



2. $(A_1 \cup A_2 \cup A_3)^c$



Countable intersections can be expressed using countable unions and taking its complements.

$$\left(\bigcup_j A_j\right)^c = \bigcap_j (A_j^c) \qquad \cdots \text{ De Morgan's law}$$

As $\mathcal{F}$ is closed under countable unions and complements,

$$\underbrace{\left(\underbrace{\bigcup_j \underbrace{A_j}_{\in\mathcal{F}}}_{\in\mathcal{F}}\right)^c}_{\in\mathcal{F}} \in \mathcal{F} \iff \bigcap_j (A_j^c) \in \mathcal{F}$$

Now, instead if we begin with A_j^c -

$$\left(\bigcup_j A_j^c\right)^c \in \mathcal{F} \iff \bigcap_j \left((A_j^c)^c\right) = \bigcap_j A_j \in \mathcal{F}$$

Thus $\mathcal{F}$ is also closed under countable intersections. $\square$