**Nagoya University, G30 program** Fall 2023

**Calculus I** Instructor : Serge Richard

**Midterm**

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**Exercise 1** *Compute the following limits:*

$$a)\ \lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}, \qquad b)\ \lim_{x\nearrow -1}\frac{x^2-x-2}{|x+1|} \qquad c)\ \lim_{x\to 0}\frac{\sin^2(x)}{x} \qquad d)\ \lim_{x\to\infty}\frac{(\mathrm{e}^x-\mathrm{e}^{-x})^2}{\mathrm{e}^{2x}+\mathrm{e}^{-2x}}.$$

**Exercise 2** *Compute the derivative of the following functions:*

1. $f:\mathbb{R}\to\mathbb{R},\ f(x)=\mathrm{e}^{\frac{1}{2}x}\sin(2x)$,
2. $g:\mathbb{R}\to\mathbb{R},\ g(x)=\sin^2\left(x^2\right)$,
3. $h:\mathbb{R}\to\mathbb{R},\ h(x)=\frac{x-1}{x^2+1}$.

**Exercise 3** *Determine the following limit:* $\lim_{x\to 1}\frac{x\ln(x)-x+1}{x\ln(x)-\ln(x)}$.

**Exercise 4** *Prove the following statement:* Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$. If $f'(x)\geq 0$ for any $x\in(a,b)$, then the function is increasing on $[a,b]$, and strictly increasing if $f'(x)>0$.

**Exercise 5** *Consider the function* $\tanh:\mathbb{R}\to\mathbb{R}$ *given by* $\tanh(x):=\frac{\sinh(x)}{\cosh(x)}=\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}}$. *Determine the range of this function, and show that it is invertible on its range. Compute the derivative of its inverse.*

**Exercise 6** *Consider the curve in* $\mathbb{R}^2$ *defined by the relation* $F(x,y)=0$ *for*

$$F(x,y)=3x^3y-y^4+5x^2+5.$$

1. *Show that the point* $(1,2)$ *belong to the curve, and find the slope of the tangent at the point* $(1,2)$,
2. *Show that the points* $\left(0,-\sqrt[4]{5}\right)$ *and* $\left(0,\sqrt[4]{5}\right)$ *belong to the curve and determine the equation of the tangent to the curve at these points.*
3. *For the 2 points* $\left(0,-\sqrt[4]{5}\right)$ *and* $\left(0,\sqrt[4]{5}\right)$, *indicate the position of the curve with respect to the tangent.*

# Midterm

Total : 26 pts

## Exercise 1  4 pts

a) $\lim\limits_{h\to 0} \dfrac{\sqrt{1+h}-1}{h} = \lim\limits_{h\to 0} \dfrac{(1+h)-1}{h(\sqrt{1+h}+1)} = \lim\limits_{h\to 0} \dfrac{1}{\sqrt{1+h}+1} = \underline{\underline{\dfrac{1}{2}}}.$

b) $\lim\limits_{x\nearrow -1} \dfrac{x^2-x-2}{|x+1|} = \lim\limits_{x\nearrow -1} \dfrac{(x+1)(x-2)}{-(x+1)} = -(-1-2) = \underline{\underline{3}}.$

c) $\lim\limits_{x\to 0} \dfrac{\sin^2(x)}{x} = \lim\limits_{x\to 0} \dfrac{\sin(x)}{x} \sin(x) = 1 \cdot 0 = \underline{\underline{0}}.$

d) $\lim\limits_{x\to\infty} \dfrac{(e^x+e^{-x})^2}{e^{2x}+e^{-2x}} = \lim\limits_{x\to\infty} \dfrac{e^{2x}+2+e^{-2x}}{e^{2x}+e^{-2x}}$

$= \lim\limits_{x\to\infty} \dfrac{e^{2x}(1+2e^{-2x}+e^{-4x})}{e^{2x}(1+e^{-4x})} = \underline{\underline{1}}.$

## Exercise 2  6 pts

1. $f'(x) = \frac{1}{2} e^{\frac{1}{2}x} \sin(2x) + e^{\frac{1}{2}x} \cos(2x)\, 2$

$= \underline{\underline{e^{\frac{1}{2}x}\left(\frac{1}{2}\sin(2x) + 2\cos(2x)\right)}}.$

2. $g'(x) = 2\sin(x^2)\cos(x^2)\, 2x = \underline{\underline{4x\sin(x^2)\cos(x^2)}}.$

3. $h'(x) = \dfrac{x^2+1-2x(x-1)}{(x^2+1)^2} = \underline{\underline{\dfrac{-x^2+2x+1}{(x^2+1)^2}}}.$

## Exercise 3 3 pts

All conditions for applying L'Hospital's rule are satisfied (since $\ln(1) = 0$), then one has

$$\lim_{x \to 1} \frac{x\ln(x) - x + 1}{x\ln(x) - \ln(x)} \overset{\text{Hôpital}}{=} \lim_{x \to 1} \frac{\ln(x) + x\frac{1}{x} - 1}{\ln(x) + x\frac{1}{x} - \frac{1}{x}}$$

$$= \lim_{x \to 1} \frac{\ln(x)}{\ln(x) + 1 - \frac{1}{x}} \underset{\text{Hôpital}}{=} \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \underline{\underline{\frac{1}{2}}}.$$

## Exercise 4 3 pts

Choose any $\alpha, \beta \in [a, b]$ with $\alpha < \beta$. Then $f : [\alpha, \beta] \to \mathbb{R}$ 1 (restriction of $f$ on $[\alpha, \beta]$) is continuous and differentiable on $(\alpha, \beta)$. We can apply the mean value thm 1 : $\exists\, \gamma \in (\alpha, \beta)$ s.t.

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = f'(\gamma) \iff f(\beta) = f(\alpha) + f'(\gamma) \underbrace{(\beta - \alpha)}_{> 0}.$$

If $f'(\gamma) \geq 0$, then $f(\beta) \geq f(\alpha)$. If $f'(\gamma) > 0$, then $f(\beta) > f(\alpha)$. Since $\alpha, \beta$ are arbitrary, $f$ is increasing or strictly increasing on $[a, b]$. 1

Exercise 5 3 pts

One has $\lim_{x \to -\infty} \tanh(x) = \lim_{x \to -\infty} \frac{\cancel{e^{-x}}(e^{2x}-1)}{\cancel{e^{-x}}(e^{2x}+1)} = -1$,

and $\lim_{x \to \infty} \tanh(x) = \lim_{x \to \infty} \frac{\cancel{e^{x}}(1-e^{-2x})}{\cancel{e^{x}}(1+e^{-2x})} = 1$. 1

For the derivative of tanh, one has

$\tanh' = \frac{\cosh^2 - \sinh^2}{\cosh^2} \overset{*}{=} \frac{1}{\cosh^2} > 0$. Thus, the

function is strictly increasing, with range $(-1,1)$. 1

$\Rightarrow$ tanh has an inverse, defined on $(-1,1)$,

denoted by arctanh. Observe from $*$ that

$\tanh'(x) = 1 - \tanh^2(x)$. Thus, for $y \in (-1,1)$:

$$\operatorname{arctanh}(y)' = \frac{1}{\tanh'(\operatorname{arctanh}(y))} = \frac{1}{1-\tanh(\operatorname{arctanh}(y))^2}$$

$$= \frac{1}{1-y^2}.$$ 1

7 pts

Exercise 6 $\quad F(x,y) = 3x^3 y - y^4 + 5x^2 + 5$

1. $F(1,2) = 3\cdot 1 \cdot 2 - 16 + 5 + 5 = 0 \Rightarrow (1,2)$ belongs to the curve. 1 Suppose that locally, one has $y = y(x)$ which satisfies $F(x, y(x)) = 0$ for $x$ belonging to a small interval of $\mathbb{R}$.

3 pts

Then $\frac{d}{dx} F(x, y(x)) = 0$

$\Leftrightarrow 9x^2 y(x) + 3x^3 y'(x) - 4y^3(x)\, y'(x) + 10x = 0$

$\Leftrightarrow y'(x)\left(3x^2 - 4y^3(x)\right) = -9x^2 y(x) - 10x$

$\Leftrightarrow \underline{\underline{y'(x) = -\frac{x(9xy(x)+10)}{3x^2 - 4y^3(x)}}}$ $\circledast$. 1

valid as long as the denominator is not 0

For $x = 1$, $y(1) = 2$, one gets

$$y'(1) = -\frac{18+10}{3-32} = \frac{28}{29}$$

The slope of the tangent is $\underline{\underline{\frac{28}{29}}}$. 1

2. Clearly $F(0, \pm\sqrt[4]{5}) = -(\pm\sqrt[4]{5})^4 + 5 = 0$

2pts

$\Rightarrow (0, \pm\sqrt[4]{5})$ belong to the curve. 1

By evaluating ⊛ at 0, one firstly observe that the denominator is not 0 $(3\,0^2 - 4(\sqrt[4]{5})^3 \neq 0)$, and then $y'(0) = 0$

$\Rightarrow$ the tangent at $(0, \pm\sqrt[4]{5})$ are the horizontal lines passing through $(0, \pm\sqrt[4]{5})$. 1

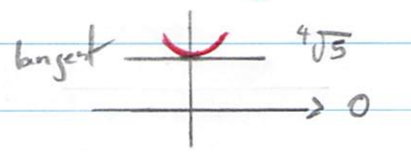
**3.** We need to compute $y''$ and evaluate it at $x = 0$. However, since $y'(0) = 0$, all terms with $y'(0)$ can be directly disregarded.

2pts

$$y''(0) = \frac{-1(3\cdot 0\, y(0) + 10)(-4y^3(0)) - 0 - 0}{(-4y^3(0))^2}$$

$$= \frac{40}{16}\,\frac{y^3(0)}{y^6(0)} = \frac{5}{2}\,\frac{1}{y(0)^3}.$$

At $(0, \sqrt[4]{5})$, $y''(0) > 0$ $\Rightarrow$




at $(0, -\sqrt[4]{5})$, $y''(0) < 0$ $\Rightarrow$