Nagoya University, G30 program Fall 2023

Calculus I Instructor : Serge Richard

Final examination Be useful !

Exercise 1 *Compute the derivative of the following functions:*

$$(i)\ x \mapsto \frac{x^2-1}{x^2+1} \qquad (ii)\ x \mapsto x\sin(1/x) \qquad (iii)\ x \mapsto \ln\Big(\ln\big(\ln(x)\big)\Big), \quad x > \mathrm{e}.$$

Exercise 2 *Determine the following limit:* $\lim_{x\to 1} \frac{x\ln(x)-x+1}{x\ln(x)-\ln(x)}$.

Exercise 3 *Compute the following integrals:*

$$(i)\ \int_1^2 \frac{1}{x(x+1)}\,\mathrm{d}x \qquad (ii)\ \int \ln(x)\,\mathrm{d}x \qquad (iii)\ \int_0^1 \sqrt{1-x^2}\,\mathrm{d}x.$$

Exercise 4 *Consider the function* $f:\mathbb{R}\to\mathbb{R}$ *with* $f(x) = \begin{cases} \mathrm{e}^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

1. *Sketch this function,*
2. *Compute its Taylor polynomial* $p_2(x, x_0)$ *at* x_0 *for any* $x_0 \neq 0$,
3. *Compute the polynomial* $p_2(x, 0)$ *by considering* $\lim_{x_0\to 0} p_2(x, x_0)$*, justify your computation,*
4. *What do you conclude about the Taylor expansion of* f *at* 0 *?*

Exercise 5 *Let* $\alpha > 0$ *and set* $f : [0, 1] \ni x \mapsto \alpha x \ \in \mathbb{R}$. *Consider the volume of revolution generated by the rotation of* $\big\{\big(x, f(x)\big) \mid x \in [0, 1]\big\}$ *around the* x*-axis.*

1. *With Riemann sums, determine the volume* V_α *of this solid (this volume depends on* α*),* you can use that $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.
2. *Compute the lateral surface of this solid with the formula*
$$S_\alpha = 2\pi \int_0^1 f(x)\sqrt{1 + f'(x)^2}\ \mathrm{d}x\ ,$$
3. *Compute the ratio* V_α/S_α *and represent this ratio as a function of* α. *In particular compute the limit* $\alpha \searrow 0$ *and the limit* $\alpha \nearrow \infty$.

Exercise 6 *Consider the sequence* $(a_j)_{j=1}^{\infty}$ *with* $a_j = \frac{1}{j(j+1)}$.

1. *Is the corresponding series convergent, justify your answer,*
2. *Determine the radius of convergence* R *of the corresponding power series* $\sum_{j=1}^{\infty} a_j x^j$,
3. *Determine if the two series* $\sum_{j=1}^{\infty} a_j R^j$ *and* $\sum_{j=1}^{\infty} a_j(-R)^j$ *converge, and if so what is the value of these series. The following equality can be used:*
$$\sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} = \ln(2).$$

Final

Total 32 pts

Exercise 1 6 pts

2 1) $\left(\dfrac{x^2-1}{x^2+1}\right)' = \dfrac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2} = \underline{\underline{\dfrac{4x}{(x^2+1)^2}}}$.

2 2) $(x\sin(1/x))' = \sin(1/x) + x\cos(1/x)(-1/x^2)$

$= \underline{\underline{\sin(1/x) - \frac{1}{x}\cos(1/x)}}$.

2 3) $\ln(\ln(\ln(x)))' = \underline{\underline{\dfrac{1}{\ln(\ln(x))}\,\dfrac{1}{\ln(x)}\,\dfrac{1}{x}}}$.

Exercise 2 (see midterm) 3 pts

L' Hôpital's rule can be aplied 2 times

$$\lim_{x\to 1} \frac{x\ln(x)-x+1}{x\ln(x)-\ln(x)} = \lim_{x\to 1}\frac{\ln(x)+1-1}{\ln(x)+1-1/x}$$

$$= \lim_{x\to 1}\frac{\ln(x)}{\ln(x)+1-1/x} = \lim_{x\to 1}\frac{1/x}{1/x+1/x^2} = \underline{\underline{\frac{1}{2}}}.$$

Exercise 3 6 pts

2 1) $\int_1^2 \frac{1}{x(x+1)}\,dx = \int_1^2 \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

$= \ln(2) - \ln(1) - (\ln(3) - \ln(2))$

$= 2\ln(2) - \ln(3) = \ln(4) - \ln(3) = \underline{\underline{\ln\left(\frac{4}{3}\right)}}.$

2 2) $\int 1 \cdot \ln(x) = x\ln(x) - \int x \frac{1}{x}\,dx$

$= x\ln(x) - x = \underline{\underline{x(\ln(x) - 1)}}.$

2 3) $\int_0^1 \sqrt{1-x^2}\,dx$ set $\sin(t) = x$

$\cos(t)\,dt = dx$

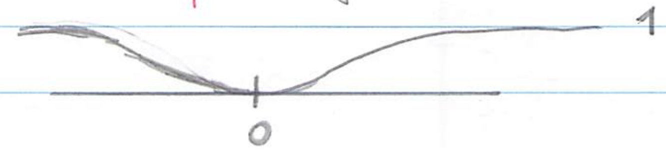
$= \int_0^{\pi/2} \sqrt{1-\sin^2(t)}\ \cos(t)\,dt$

$= \int_0^{\pi/2} \cos^2(t)\,dt = \int_0^{\pi/2} \frac{1+\cos(2t)}{2}\,dt$

$= \frac{1}{2}\frac{\pi}{2} + \frac{1}{2}\int_0^{\pi/2} \cos(2t)\,dt = \frac{\pi}{4} + \frac{1}{4}\sin(2t)\Big|_0^{\pi/2}$

$= \underline{\underline{\frac{\pi}{4}}}.$

Exercise 4 7pts $f(x) = e^{-1/x^2}$, $x \in \mathbb{R}$

1 1) 

the function is even.

3 2) $f(x_0) = e^{-1/x_0^2}$

$$f'(x) = e^{-x^{-2}}(+2x^{-3}) = 2x^{-3}e^{-x^{-2}}$$

$$\Rightarrow f'(x_0) = 2x_0^{-3}e^{-x_0^{-2}}.$$

$$f''(x) = -6x^{-4}e^{-x^{-2}} + 4(x^{-3})^2 e^{-x^{-2}}$$

$$\Rightarrow f''(x_0) = \left(-6x_0^{-4} + 4x_0^{-6}\right)e^{-x_0^{-2}}.$$

$$\Rightarrow P_2(x, x_0) = e^{-x_0^{-2}}\left(1 + \frac{2}{x_0^3}(x-x_0) + \frac{1}{2}\left(-\frac{6}{x_0^4} + \frac{4}{x_0^6}\right)(x-x_0)^2\right)$$

$$= e^{-x_0^{-2}}\left(1 + \frac{2}{x_0^3}(x-x_0) + \left(-\frac{3}{x_0^4} + \frac{2}{x_0^6}\right)(x-x_0)^2\right).$$

2. 3) Observe that $\lim_{x_0 \searrow 0} \frac{e^{-x_0^{-2}}}{x_0^n} = \lim_{y \to \infty} \frac{y^n}{e^{y^2}} = 0$ set $y = \frac{1}{x_0}$

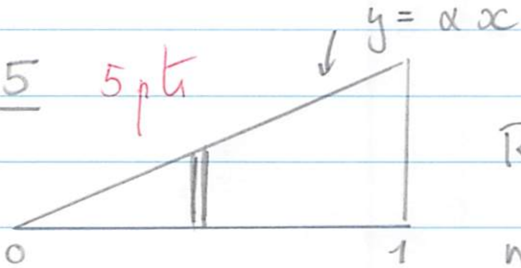
and similarly $\lim_{x_0 \nearrow 0} \frac{e^{-x_0^{-2}}}{x_0^n} = \lim_{y \to -\infty} \frac{y^n}{e^{y^2}} = 0$

for any $n \in \mathbb{N}$. set $y = 1/x_0$

Thus $\lim_{x_0 \to 0} P_2(x, x_0) = 0$. the 0-function

1 4) The function $x \mapsto e^{-x^{-2}}$ does not admit a Taylor expansion at 0.

Exercise 5 5 pts



2 1) Regular partition in n subintervals :

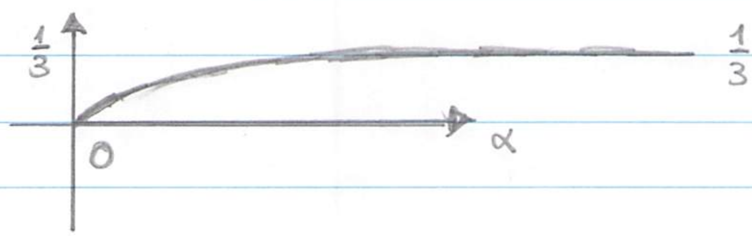
$$R_{right}(f,P) = \sum_{j=1}^{n} \pi \frac{1}{n} \left(\alpha \frac{j}{n}\right)^2 = \pi \alpha^2 \frac{1}{n^3} \sum_{j=1}^{n} j^2$$

$$= \pi \alpha^2 \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \xrightarrow{n \to \infty} \underline{\frac{\pi \alpha^2}{3}} =: V_\alpha$$

1 2) $S_\alpha = 2\pi \int_0^1 \alpha x \sqrt{1+\alpha^2}\, dx$

$$= \underline{\pi \alpha \sqrt{1+\alpha^2}} .$$

2 3) $\dfrac{V_\alpha}{S_\alpha} = \dfrac{\pi \alpha^2}{3\pi\alpha\sqrt{1+\alpha^2}} = \dfrac{1}{3} \dfrac{\alpha}{\sqrt{1+\alpha^2}}$



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Exercise 6 5 pts $a_j = \frac{1}{j(j+1)}$

2 1) One has $a_j = \frac{1}{j(j+1)} \leq \frac{1}{j^2}$ and $\sum_{j=1}^{\infty} \frac{1}{j^2} < \infty$, as seen during the lecture, or by the integral test. By comparison lemma, $\sum_{j=1}^{\infty} \frac{1}{j(j+1)} < \infty$.

Thus the series converges.

1 2) By the ratio test: $\frac{a_{j+1} r^{j+1}}{a_j r^j} = \frac{j(j+1)}{(j+1)(j+2)} r$

$= \frac{j}{j+2} r$. Since $\frac{j}{j+2} \xrightarrow{j \to \infty} 1$, the radius of convergence is $R = 1$.

2 3) One has $\sum_{j=1}^{\infty} a_j 1^j = \sum_{j=1}^{\infty} \frac{1}{j(j+1)} = \sum_{j=1}^{\infty} \left(\frac{1}{j} - \frac{1}{j+1} \right)$

$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} \dots = \underline{1}.$

One has $\sum_{j=1}^{\infty} a_j (-1)^j = \sum_{j=1}^{\infty} \frac{(-1)^j}{j} - \frac{(-1)^j}{j+1}$

$= - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} - \sum_{j=1}^{\infty} \frac{(-1)^{j+2}}{j+1} =$

$= - \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} - \left(\sum_{k=2}^{\infty} \frac{(-1)^{k+1}}{k} + 1 - 1 \right)$

$= \underline{-2 \ln(2) + 1}.$