
Homework 7

Exercise 1 Prove the following properties of the function \ln :

(i) $\ln(x)' = \frac{1}{x}$ for any $x \in (0, \infty)$,

(ii) $\ln(xy) = \ln(x) + \ln(y)$ for any $x, y \in (0, \infty)$,

(iii) $\ln(x^q) = q \ln(x)$ for any $x \in (0, \infty)$ and $q \in \mathbb{Q}$.

Exercise 2 Let us set $\varepsilon := e^1 = 2.718\dots$. Check that $\ln(\varepsilon) = 1$ and that $\varepsilon^x = e^x$.

Exercise 3 Compute the derivative of the following functions:

$$f : \mathbb{R} \ni x \mapsto a^x \in \mathbb{R} \text{ for any } a > 0, \quad g : \mathbb{R}_+^* \ni x \mapsto x^x \in \mathbb{R}.$$

Exercise 4 Compute the following limits:

$$a) \lim_{x \rightarrow 0_+} x \ln(x), \quad b) \lim_{x \rightarrow 0_+} x^x, \quad c) \lim_{x \rightarrow +\infty} \frac{\ln(x)}{x}, \quad d) \lim_{x \rightarrow +\infty} x^{1/x}.$$

What can you say for $\lim_{x \rightarrow 0_+} x^r \ln(x)$ for any $r > 0$?

Exercise 5 Compute the following limits:

$$a) \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}, \quad b) \lim_{x \rightarrow 0} (1+x)^{1/x}, \quad c) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x, \quad d) \lim_{x \rightarrow \infty} \left(1 + \frac{r}{x}\right)^x \text{ for any } r > 0.$$

Exercise 6 Compute $\lim_{x \rightarrow 0_+} (1 + \sin(4x))^{\cot(x)}$ with $\cot(x) = \frac{1}{\tan(x)}$.

Exercise 7 Show that $\tanh(y)^{-1} = \frac{1}{2} \ln \left(\frac{1+y}{1-y} \right)$ for any $y \in (-1, 1)$.

Exercise 8 Differentiate the function $\mathbb{R}_+ \ni x \mapsto \frac{x^{3/4} \sqrt{x^2+1}}{(3x+2)^5} \in \mathbb{R}_+$.