Report

Li Yucheng

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1 Proof of exercise 4.4.4

We consider a complex semi-simple Lie algebra L endowed with its standard basis, and let L_0 denote its Cartan subalgebra. Let also $(\mathscr{V}, h)$ be a finite dimensional **irreducible** representation. We denote by μ_{max} the maximal weight, once $\mathbb{R}^{d_0}$ is endowed with the lexicographic order, starting from the last component. Clearly, if α is a positive root, then $L_{\mu_{max}+\alpha} = 0$, otherwise one would get a contradiction. Equivalently. $\mathfrak{E}_\alpha v_{max} = 0$ for the maximal weight vector v_{max}. By collecting the information obtained some far, the following exercise is rather instructing:

$$\begin{aligned}
\mathscr{V} =& Span\{v_{max}, \mathfrak{E}_\alpha v_{max}, \mathfrak{E}_\alpha \mathfrak{E}_\beta v_{max}, ...|\alpha, \beta ... \in \mathscr{R}_-\} && (1)\\
=& Span\{v_{max}, \mathfrak{E}_{-\alpha} v_{max}, \mathfrak{E}_{-\alpha} \mathfrak{E}_{-\beta} v_{max}, ...|\alpha, \beta, ... \in \{simple \quad roots\}\} && (2)\\
=& Span\{v, \mathfrak{E}_\alpha v, \mathfrak{E}_\alpha \mathfrak{E}_\beta v, ...|\alpha, \beta ... \in \mathscr{R}\}, && (3)
\end{aligned}$$

where v is an arbitrary weight vector in the last row.

Proof: For the first row, from proposition 4.4.1 and the fact that $\alpha, \beta ... \in \mathscr{R}_-$, we can see that the vectors that span the space are weight vectors corresponding to weights smaller than μ_{max}, which are allowed to exist. Then, these vectors corresponding to different weights are linearly independent to each other. If we act an arbitrary $h(H_i)$ and $h(E_i)$(here I write like this for consistency) on these vectors, we can find that this spanned space is invariant under the representation, because these vectors are already eigenvectors of $h(H_i)$, and acting $h(E_i)$ on them only produces another vector in the row. Since this set is obviously not empty and the representation is irreducible, we have to conclude that this span is $\mathscr{V}$ itself.

For the second row, recalling the definition of simple root in 4.3.10, we can see that the second row rules out the vectors which may correspond to the same weights in the first row, only keeps the "effective" vectors, which span the same space $\mathscr{V}$ as the first row.

For the last row, the thinking is the same as the first and the second row.