# Report

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## 1 Proof of Exercise 2.3.2.

Let $v$ be an arbitrary vector in space $V$. Then we obviously have a conclusion that $dim(U(G)v) \leq |G|$, that is, the dimension of subspace formed by $U(G)v$ should not be larger than the cardinality of group G. Notice that $U(G)v$ is an invariant subspace, because $U(G)U(G)v = U(GG)v = U(G)v$. According to the definition of irreducibility, this $U(G)v$ is not 0, so it must be the space $V$ itself. Thus we have $dim(V) = dim(U(G)v) \leq |G|$.

## 2 Proof of Proposition 2.3.9

First, let us define the *subrepresentation* (according to Serre's book).

***Definition***:Let $U : G \mapsto \mathscr{L}(V)$ be a linear representation and let $W$ be a subspace of $V$. Suppose that $W$ is invariant under the action of $G$. The restriction $U(G)$ of $U(G)$ to $W$ is then an isomorphism of $W$ onto itself, and we have $U(a)U(b) = U(ab)$, for any $a, b$ in $W$.Thus $U^{\cdot}G \mapsto \mathscr{L}(W)$ is a linear representation of $G$ in $W$; $W$ is said to be a *subrepresentation* of $V$.

***Proposition***: Let $G$ be a finite group and assume that $G_0$ is an Abelian subgroup of $G$. Then any irreducible representation of $G$ is of dimension at most $|G| \,/\, |G_0|$.

***Proof***: Let $U : G \mapsto \mathscr{L}(V)$ be an irreducible representation of $G$. Now we restrict our mind to the subgroup $G_0$, and by the same representation only acting on $G_0$, we still have $U : G_0 \mapsto \mathscr{L}(V)$. Let $W \subset V$ be an irreducible subrepresentation of $U$ acting on $G_0$. By Corollary 2.3.8, we have $dim(W) = 1$. Just like what we did in the proof of Exercise 2.3.2, the space $U(G)W$ is obviously invariant. Since $U$ is irreducible, $U(G)W$ must be $V$ itself. Notice that for any element $s \in G$, $U(sG_0)W = U(s)U(G_0)W = U(s)W$, where $U(s)W$ is an arbitrary vector in $V$. We can find that through the representation map $U$, we have defined an equivalence relation between $sG_0$ and $s$, or a left coset. This means that $U(sG_0)W$ and $U(s)$ denotes the same vector in $V$. Thus, the dimension of $V$ means the number of distinct cosets of $G_0$ in $G$. So one should naturally get that $dim(V) \leq |G| \,/\, |G_0| \times dim(W) = |G| \,/\, |G_0|$, where $|G| \,/\, |G_0|$ is defined as the number of cosets.