# The trace and the norm of the tensor product of linear operators

NGUYEN Tue Tai / 062201848

November 16, 2022

## Exercise 2.5.3

**1.** *Consider $A_1 \in \mathcal{B}(\mathcal{H}_1)$ and $A_2 \in \mathcal{B}(\mathcal{H}_2)$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are finit dimensional, then*

$$\mathrm{Tr}\,(A_1 \otimes A_2) = \mathrm{Tr}(A_1)\,\mathrm{Tr}(A_2).$$

*Proof.* Let $\{e_j^1\}_j$ and $\{e_k^2\}_k$ be orthonormal bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then, the set $\{e_j^1 \otimes e_k^2\}_{j,k}$ is an orthonormal basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Hence, we have

$$\begin{aligned}
\mathrm{Tr}\,(A_1 \otimes A_2) &= \sum_{j,k} \left\langle e_j^1 \otimes e_k^2, (A_1 \otimes A_2)\left(e_j^1 \otimes e_k^2\right)\right\rangle \\
&= \sum_{j,k} \left\langle e_j^1 \otimes e_k^2, (A_1 e_j^1) \otimes (A_2 e_k^2)\right\rangle \\
&= \sum_{j,k} \left\langle e_j^1, A_1 e_j^1\right\rangle \left\langle e_k^2, A_2 e_k^2\right\rangle \\
&= \sum_{j} \left\langle e_j^1, A_1 e_j^1\right\rangle \sum_{k} \left\langle e_k^2, A_2 e_k^2\right\rangle \\
&= \mathrm{Tr}(A_1)\,\mathrm{Tr}(A_2). \qquad \blacksquare
\end{aligned}$$

**2.** *Show that $\|A_1 \otimes A_2\| = \|A_1\|\|A_2\|$.*

*Proof.* For any $f \in \mathcal{H}_1 \otimes \mathcal{H}_2$, it can be represented as

$$f = \sum_{j,k} \lambda_{jk}(e_j^1 \otimes e_k^2), \qquad \text{with } \lambda_{jk} \in \mathbb{C}.$$

We can also write it in another form.

$$f = \sum_j e_j^1 \otimes \left(\sum_k \lambda_{jk} e_k^2\right) = \sum_j e_j^1 \otimes f_j^2,$$

Where

$$f_j^2 = \sum_k \lambda_{jk} e_k^2.$$

Consider the norm of $f$.

$$\begin{aligned}
\|f\|^2 = \langle f, f\rangle &= \left\langle \sum_j e_j^1 \otimes f_j^2, \sum_k e_k^1 \otimes f_k^2 \right\rangle \\
&= \sum_{j,k} \left\langle e_j^1 \otimes f_j^2, e_k^1 \otimes f_k^2\right\rangle \\
&= \sum_{j,k} \left\langle e_j^1, e_k^1\right\rangle \left\langle f_j^2, f_k^2\right\rangle \\
&= \sum_{j,k} \delta_{jk} \left\langle f_j^2, f_k^2\right\rangle \\
&= \sum_j \left\langle f_j^2, f_j^2\right\rangle = \sum_j \left\|f_j^2\right\|^2.
\end{aligned}$$

Consider the operator $\mathbb{1}_{\mathcal{H}_1} \otimes A_2$. With the same calculation as above, we have

$$\left\| \left( \mathbb{1}_{\mathcal{H}_1} \otimes A_2 \right) f \right\|^2 = \left\| \left( \mathbb{1}_{\mathcal{H}_1} \otimes A_2 \right) \sum_j e_j^1 \otimes f_j^2 \right\|^2 = \left\| \sum_j e_j^1 \otimes \left( A_2 f_j^2 \right) \right\|^2 = \sum_j \left\| A_2 f_j^2 \right\|^2.$$

From the definition of the norm of operators, we have $\left\| A_2 f_j^2 \right\| \leq \|A_2\| \left\| f_j^2 \right\|$. Hence, we have

$$\left\| \left( \mathbb{1}_{\mathcal{H}_1} \otimes A_2 \right) f \right\|^2 \leq \sum_j \|A_2\|^2 \left\| f_j^2 \right\|^2 = \|A_2\|^2 \sum_j \left\| f_j^2 \right\|^2 = \|A_2\|^2 \|f\|^2$$

Thus,

$$\|\mathbb{1}_{\mathcal{H}_1} \otimes A_2\| = \sup_{f \in \mathcal{H}_1 \otimes \mathcal{H}_2} \frac{\left\| \left( \mathbb{1}_{\mathcal{H}_1} \otimes A_2 \right) f \right\|}{\|f\|} \leq \|A_2\|$$

With similar calculation, we can also show that $\|A_1 \otimes \mathbb{1}_{\mathcal{H}_2}\| \leq \|A_1\|$.
Also, by using the property $\|AB\| \leq \|A\| \|B\|$, we have

$$\|A_1 \otimes A_2\| = \left\| \left( A_1 \otimes \mathbb{1}_{\mathcal{H}_2} \right) \left( \mathbb{1}_{\mathcal{H}_1} \otimes A_2 \right) \right\| \leq \|A_1 \otimes \mathbb{1}_{\mathcal{H}_2}\| \|\mathbb{1}_{\mathcal{H}_1} \otimes A_2\| \leq \|A_1\| \|A_2\|.$$

Furthermore, let $\mathcal{H}_0 \subset \mathcal{H}_1 \otimes \mathcal{H}_2$ be the subset of all elements that can be expressed as $f_1 \otimes f_2$, with $f_1 \in \mathcal{H}_1$, $f_2 \in \mathcal{H}_2$. Hence, we have

$$\begin{aligned} \|A_1 \otimes A_2\| = \sup_{f \in \mathcal{H}_1 \otimes \mathcal{H}_2} \frac{\|(A_1 \otimes A_2) f\|}{\|f\|} \geq \sup_{f \in \mathcal{H}_0} \frac{\|(A_1 \otimes A_2) f\|}{\|f\|} &= \sup_{f_1 \otimes f_2 \in \mathcal{H}_0} \frac{\|(A_1 \otimes A_2)(f_1 \otimes f_2)\|}{\|f_1 \otimes f_2\|} \\ &= \sup_{f_1 \otimes f_2 \in \mathcal{H}_0} \frac{\|(A_1 f_1) \otimes (A_2 f_2)\|}{\|f_1 \otimes f_2\|} \end{aligned}$$

On the other hand, we have

$$\|f_1 \otimes f_2\| = \sqrt{\langle f_1 \otimes f_2, f_1 \otimes f_2 \rangle} = \sqrt{\langle f_1, f_1 \rangle \langle f_2, f_2 \rangle} = \sqrt{\langle f_1, f_1 \rangle} \sqrt{\langle f_2, f_2 \rangle} = \|f_1\| \|f_2\|.$$

Hence,

$$\|A_1 \otimes A_2\| \geq \sup_{f_1 \otimes f_2 \in \mathcal{H}_0} \frac{\|A_1 f_1\| \|A_2 f_2\|}{\|f_1\| \|f_2\|}.$$

Because $\|A_1 f_1\| / \|f_1\|$ and $\|A_2 f_2\| / \|f_2\|$ are positive real numbers, we have

$$\|A_1 \otimes A_2\| \geq \sup_{f_1 \in \mathcal{H}_1} \frac{\|A_1 f_1\|}{\|f_1\|} \sup_{f_2 \in \mathcal{H}_2} \frac{\|A_2 f_2\|}{\|f_2\|} = \|A_1\| \|A_2\|.$$

Thus, we have already proved that $\|A_1 \otimes A_2\| \leq \|A_1\| \|A_2\|$ and $\|A_1 \otimes A_2\| \geq \|A_1\| \|A_2\|$, which implies that $\|A_1 \otimes A_2\| = \|A_1\| \|A_2\|$. ■