On the Wigner-Eckart Theorem

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1 Selection rules

Let us summarize the main ideas of the selection rule. The definitions are given in [1] and the proof is given in [2]. Also, for simplicity, we will work with finite groups and finite dimensional representations.

For a unitary representation $(\mathcal{H}, U)$, we can decompose it into irreducible representations

$$\mathcal{H} = \bigoplus \nu_j \mathcal{H}^{(j)} \quad , \quad U = \bigoplus \nu_j U^{(j)},$$

and we also have the representation $(\mathcal{B}(\mathcal{H}), \mathcal{U})$ which can be decomposed into irreducible representations

$$\mathcal{B}(\mathcal{H}) = \bigoplus \mu_l \mathcal{L}^{(l)} \quad , \quad \mathcal{U} = \bigoplus \mu_l \mathcal{U}^{(l)}.$$

Focusing on the equivalent class $\eta^{(l)}$ containing $(\mathcal{L}^{(l)}, \mathcal{U}^{(l)})$, there exists an irreducible representation $(\mathcal{H}^{(l)}, U^{(l)})$, and we denote the similarity transformation between them by

$$\tau_l : \mathcal{H}^{(l)} \to \mathcal{L}^{(l)}.$$

Then, for any $f_l \in \mathcal{H}^{(l)}$, $f_k \in \mathcal{H}_m^{(k)}$ where $m \in \{1, ..., \nu_k\}$, and $f_j \in \mathcal{H}_n^{(j)}$ where $n \in \{1, ..., \nu_j\}$, one has

$$\langle f_k, \tau_l(f_l) f_j \rangle = 0$$

unless there exists a representation of class $\eta^{(k)}$ in the decomposition of the representation $(\mathcal{H}^{(l)} \otimes \mathcal{H}^{(j)}, U^{(l)} \otimes U^{(j)})$.

2 Clebsch-Gordon coefficients

Consider two general representations $(\mathcal{H}^{(l)}, U^{(l)})$ and $(\mathcal{H}^{(j)}, U^{(j)})$ where we denote the basis of $\mathcal{H}^{(l)}$ by $\{e_r^{(l)}\}_r$ and $\mathcal{H}^{(j)}$ by $\{e_s^{(j)}\}_s$. If the representation is from a single group G, the representation $(\mathcal{H}^{(l)} \otimes \mathcal{H}^{(j)}, U^{(l)} \otimes U^{(j)})$ is not irreducible in general, meaning that we can decompose it into irreducible representation

$$\mathcal{H}^{(l)} \otimes \mathcal{H}^{(j)} = \bigoplus \alpha_m \mathcal{H}^{(m)} = \bigoplus \mathcal{H}^{(m,n)} \quad , \quad U^{(l)} \otimes U^{(j)} = \bigoplus \alpha_m U^{(m)} = \bigoplus U^{(m,n)},$$

where $n \in \{1, ..., \alpha_m\}$. Denote a basis for $\mathcal{H}^{(m,n)}$ by $\{e_o^{(m,n)}\}_o$, then we can write this basis in terms of the uncoupled basis $\{e_r^{(l)} \otimes e_s^{(j)}\}_{r,s}$ by

$$e_o^{(m,n)} = \sum_{r,s} C(mno; lj)_{rs}\, e_r^{(l)} \otimes e_s^{(j)} \;{}^{1}.$$

The coefficients $C(mno; lj)_{rs}$ are called the Clebsch-Gordon coefficients. With its application in quantum mechanics, the Dirac notation is often used, in which

$$|mno\rangle_{\oplus} := e_o^{(m,n)} \quad , \quad |lr, js\rangle_{\otimes} := e_r^{(l)} \otimes e_s^{(j)},$$

$$C(mno; lj)_{rs} := \langle lr, js|mno\rangle \quad , \quad \overline{C(mno; lj)}_{rs} := \langle mno|lr, js\rangle$$

where we have denoted $\oplus$ and $\otimes$ to distinguish between the bases. Then, the equation reads

$$|mno\rangle_{\oplus} = \sum_{r,s} |lr, js\rangle_{\otimes}\, \langle lr, js|mno\rangle.$$

Let us use the Dirac notation for the Clebsch-Gordon coefficients only. This is because by using the orthonormality of $\{e_r^{(l)} \otimes e_s^{(j)}\}_{r,s}$, we get

$$\langle e_p^{(l)} \otimes e_q^{(j)}, e_o^{(m,n)}\rangle = C(mno; lj)_{pq} = \langle lp, jq|mno\rangle,$$

so the Dirac notation gives us a natural way to write down the coefficients.

Now, let us study some properties of the Clebsch-Gordon coefficients, even though we might not use all of them. We first define a linear operator on $\mathcal{H} = \mathcal{H}^{(l)} \otimes \mathcal{H}^{(j)} = \bigoplus \mathcal{H}^{(m,n)}$

$$\begin{aligned} I_{mno} :\ & \mathcal{H} \to \mathcal{H}, \\ & f \mapsto \langle e_o^{(m,n)}, f\rangle\, e_o^{(m,n)}. \end{aligned}$$

In Dirac notation, we write this as $I_{mno} = |mno\rangle\langle mno|$. By orthonormality, we have

$$I_{mno}\, e_{o'}^{(m'n')} = \langle e_o^{(m,n)}, e_{o'}^{(m'n')}\rangle\, e_o^{(m,n)} = \delta_{mm'}\delta_{nn'}\delta_{oo'}\, e_o^{(m,n)}.$$

Then, if we define

$$\begin{aligned} I_{\oplus} :\ & \mathcal{H} \to \mathcal{H}, \\ & f \mapsto \sum_{m,n,o} I_{mno}\, f, \end{aligned}$$

its action on the basis $\{e_{o'}^{(m',n')}\}_{m',n',o'}$ of $\mathcal{H}$ is given by

$$I_{\oplus} e_{o'}^{(m'n')} = \sum_{m,n,o} \delta_{mm'}\delta_{nn'}\delta_{oo'}\, e_o^{(m,n)} = e_{o'}^{(m'n')},$$

[1] The notation (m, n) also helps us differentiate between the basis with respect to $\bigoplus \mathcal{H}^{(m,n)}$ and that with respect to $\mathcal{H}^{(l)} \otimes \mathcal{H}^{(j)}$.

meaning that $I_\oplus$ acts like the identity operator on $\mathcal{H}$. We can apply this result to the basis with respect to $\mathcal{H}^{(l)} \otimes \mathcal{H}^{(j)}$ as follows

$$\begin{aligned}\langle e_p^{(l)} \otimes e_q^{(j)}, e_r^{(l)} \otimes e_s^{(j)} \rangle &= \langle e_p^{(l)} \otimes e_q^{(j)}, I_\oplus\, e_r^{(l)} \otimes e_s^{(j)} \rangle \\ &= \sum_{m,n,o} \langle e_p^{(l)} \otimes e_q^{(j)}, e_o^{(m,n)} \rangle \langle e_o^{(m,n)}, e_r^{(l)} \otimes e_s^{(j)} \rangle \\ &= \sum_{m,n,o} \langle lp, jq | mno \rangle \langle mno | lr, js \rangle,\end{aligned}$$

but since $\langle e_p^{(l)} \otimes e_q^{(j)}, e_r^{(l)} \otimes e_s^{(j)} \rangle = \delta_{pr}\delta_{qs}$, we get

$$\sum_{m,n,o} \langle lp, jq | mno \rangle \langle mno | lr, js \rangle = \delta_{pr}\delta_{qs}. \tag{1}$$

Similarly, we can define the following operators

$$\begin{aligned} I_{lr,js} :\ & \mathcal{H} \to \mathcal{H}, \\ & f \mapsto \langle e_r^{(l)} \otimes e_s^{(j)}, f \rangle\, e_r^{(l)} \otimes e_s^{(j)}, \end{aligned}$$

$$\begin{aligned} I_\otimes :\ & \mathcal{H} \to \mathcal{H}, \\ & f \mapsto \sum_{r,s} I_{lr,js}\, f = f, \end{aligned}$$

then using the similar procedure for $\langle e_o^{(m,n)}, e_{o'}^{(m',n')} \rangle$, we get

$$\sum_{r,s} \langle mno | lr, js \rangle \langle lr, js | m'n'o' \rangle = \delta_{mm'}\delta_{nn'}\delta_{oo'}. \tag{2}$$

We will mainly deal with unitary representation so let $U^{(l)}$ and $U^{(j)}$ be unitary operators. We now show that $U = U^{(l)} \otimes U^{(j)}$ is also unitary in $\mathcal{H}$ as defined previously. For brevity, we will omit the group element a from $U(a)$. Recall that U is unitary if for any $f, g \in \mathcal{H}$, one has

$$\langle Uf, Ug \rangle = \langle f, g \rangle$$

Due to the linearity of the map and the inner product, it is sufficient to consider the basis $\{e_r^{(l)} \otimes e_s^{(j)}\}_{r,s}$, i.e., for arbitrary p, q, r, s,

$$\begin{aligned}\left\langle U(e_p^{(l)} \otimes e_q^{(j)}), U(e_r^{(l)} \otimes e_s^{(j)}) \right\rangle &= \left\langle (U^{(l)} e_p^{(l)}) \otimes (U^{(j)} e_q^{(j)}), (U^{(l)} e_r^{(l)}) \otimes (U^{(j)} e_s^{(j)}) \right\rangle \\ &= \langle U^{(l)} e_p^{(l)}, U^{(l)} e_r^{(l)} \rangle\, \langle U^{(j)} e_q^{(j)}, U^{(j)} e_s^{(j)} \rangle \\ &= \langle e_p^{(l)}, e_r^{(l)} \rangle\, \langle e_q^{(j)}, e_s^{(j)} \rangle \\ &= \langle e_p^{(l)} \otimes e_q^{(j)}, e_r^{(l)} \otimes e_s^{(j)} \rangle,\end{aligned}$$

where we have used unitary property of $U^{(l)}$ and $U^{(j)}$ at the third line. Thus, U is indeed unitary. The basis $\{e_o^{(m,n)}\}_{m,n,o}$ is given by a linear combination of the uncoupled basis so the result still holds even if we consider the new basis. Using $I_\oplus$ and the fact that $U^{(m,n)}$ only acts on $\mathcal{H}^{(m,n)}$, one has

$$\begin{aligned}
U(e_r^{(l)} \otimes e_s^{(j)}) &= I_\oplus U\big(I_\oplus(e_r^{(l)} \otimes e_s^{(j)})\big) \\
&= \sum_{\substack{m,n,o,\\ m',n',o'}} \langle e_o^{(m,n)}, e_r^{(l)} \otimes e_s^{(j)}\rangle \, \langle e_{o'}^{(m',n')}, U\big(e_o^{(m,n)}\big)\rangle \, e_{o'}^{(m',n')} \\
&= \sum_{\substack{m,n,o,\\ m',n',o'}} \langle mno|lr, js\rangle \, \delta_{mm'}\delta_{nn'} U_{o'o}^{(m,n)} \, e_{o'}^{(m',n')} \\
&= \sum_{m,n,o,o'} \langle mno|lr, js\rangle \, U_{o'o}^{(m,n)} \, e_{o'}^{(m,n)}.
\end{aligned}$$

Using the expansion of $e_{o'}^{(m,n)}$ in terms of $\{e_u^{(l)} \otimes e_v^{(j)}\}_{u,v}$,

$$U(e_r^{(l)} \otimes e_s^{(j)}) = \sum_{\substack{m,n,o,o',\\ u,v}} \langle mno|lr, js\rangle \, \langle lu, jv|mno'\rangle \, U_{o'o}^{(m,n)} \, e_u^{(l)} \otimes e_v^{(j)}.$$

Then, take the inner product with $e_p^{(l)} \otimes e_q^{(j)}$

$$\begin{aligned}
\big\langle e_p^{(l)} \otimes e_q^{(j)}, U(e_r^{(l)} \otimes e_s^{(j)})\big\rangle &= \sum_{\substack{m,n,o,o',\\ u,v}} \langle mno|lr, js\rangle \, \langle lu, jv|mno'\rangle \, \langle e_p^{(l)} \otimes e_q^{(j)}, e_u^{(l)} \otimes e_v^{(j)}\rangle \, U_{o'o}^{(m,n)} \\
&= \sum_{\substack{m,n,o,o',\\ u,v}} \langle mno|lr, js\rangle \, \langle lu, jv|mno'\rangle \, \delta_{pu}\delta_{qv} U_{o'o}^{(m,n)} \\
&= \sum_{m,n,o,o'} \langle mno|lr, js\rangle \, \langle lp, jq|mno'\rangle \, U_{o'o}^{(m,n)},
\end{aligned}$$

and by using the matrix elements of $U^{(l)}$ and $U^{(j)}$, we get

$$U_{pr}^{(l)} U_{qs}^{(j)} = \sum_{m,n,o,o'} \langle mno|lr, js\rangle \, \langle lp, jq|mno'\rangle \, U_{o'o}^{(m,n)}.$$

From the lecture, if we denote $\dim(\mathcal{H}^{(m,n)}) = d^m$, the orthogonality relation gives

$$\frac{1}{|G|} \sum_{a\in G} \overline{U_{\rho'\rho}^{(m',n')}(a)} \, U_{o'o}^{(m,n)}(a) = \frac{1}{d_m} \delta_{mm'}\delta_{nn'}\delta_{o'\rho'}\delta_{o\rho}.$$

Thus,

$$\begin{aligned}
\frac{1}{|G|} \sum_{a\in G} \overline{U_{\rho'\rho}^{(m',n')}(a)} \, U_{pr}^{(l)}(a) \, U_{qs}^{(j)}(a) &= \frac{1}{d_m} \sum_{m,n,o,o'} \langle mno|lr, js\rangle \, \langle lp, jq|mno'\rangle \;\; \delta_{mm'}\delta_{nn'}\delta_{o'\rho'}\delta_{o\rho} \\
&= \frac{1}{d_{m'}} \langle m'n'\rho|lr, js\rangle \langle lp, jq|m'n'\rho'\rangle.
\end{aligned}$$

We can reindex $m' \to m$, $n' \to n$, then we have

$$\frac{1}{|G|} \sum_{a\in G} \overline{U_{\rho'\rho}^{(m,n)}(a)} \, U_{pr}^{(l)}(a) \, U_{qs}^{(j)}(a) = \frac{1}{d_m} \langle mn\rho|lr, js\rangle \langle lp, jq|mn\rho'\rangle. \tag{3}$$

3 Wigner-Eckart theorem

Let us define $\mathcal{H}^{(l)}$ and $\mathcal{H}^{(j)}$ in section 2 to be the same as that from section 1. Assume that there exists a representation of class $\eta^{(k)}$ in the decomposition of $\mathcal{H}^{(l)} \otimes \mathcal{H}^{(j)}$. In other words, there exists (m, n) in the decomposition

$$\mathcal{H}^{(l)} \otimes \mathcal{H}^{(j)} = \bigoplus \alpha_m \mathcal{H}^{(m)} = \bigoplus \mathcal{H}^{(m,n)}$$

such that $\eta^{(m,n)} = \eta^{(k)}$. Then, it is possible to calculate the inner product given in section 1, and the result is called the Wigner-Eckart theorem. Because of the linearity of the maps in the expression, it is sufficient to consider the bases $\{e_o^{(k)}\}_o, \{e_r^{(l)}\}_r, \{e_s^{(j)}\}_s$ of $\mathcal{H}^{(k)}, \mathcal{H}^{(l)}, \mathcal{H}^{(j)}$, respectively. We will omit the group element a for now, then one has

$$\begin{aligned}
\left\langle e_o^{(k)}, \tau_l(e_r^{(l)}) e_s^{(j)} \right\rangle &= \left\langle U e_o^{(k)}, U \tau_l(e_r^{(l)}) e_s^{(j)} \right\rangle && \text{(by unitary)} \\
&= \left\langle U^{(k)} e_o^{(k)}, U \tau_l(e_r^{(l)}) U^{-1} U e_s^{(j)} \right\rangle && \text{(adding } I = U^{-1}U\text{)} \\
&= \left\langle U^{(k)} e_o^{(k)}, \tau_l(U^{(l)} e_r^{(l)}) U^{(j)} e_s^{(j)} \right\rangle && \text{(by definition).}
\end{aligned}$$

By adding the appropriate identity operator and using orthonormality,

$$\begin{aligned}
U^{(k)} e_o^{(k)} &= I U^{(k)} e_o^{(k)} \\
&= \sum_{k', o'} \langle e_{o'}^{(k')}, U^{(k)} e_o^{(k)} \rangle e_{o'}^{(k')} \\
&= \sum_{k', o'} \delta_{k'k} U_{o'o}^{(k)} e_{o'}^{(k')} \\
&= \sum_{o'} U_{o'o}^{(k)} e_{o'}^{(k)},
\end{aligned}$$

$$\begin{aligned}
\tau_l(U^{(l)} e_r^{(l)}) &= \tau_l(I U^{(l)} e_r^{(l)}) \\
&= \sum_{l', r'} \langle e_{r'}^{(l')}, U^{(l)} e_r^{(l)} \rangle \; \tau_l(e_{r'}^{(l')}) \\
&= \sum_{l', r'} \delta_{l'l} U_{r'r}^{(l)} \tau_l(e_{r'}^{(l')}) \\
&= \sum_{r'} U_{r'r}^{(l)} \tau_l(e_{r'}^{(l)})
\end{aligned}$$

$$\begin{aligned}
U^{(j)} e_s^{(j)} &= I U^{(j)} e_s^{(j)} \\
&= \sum_{j', s'} \langle e_{s'}^{(j')}, U^{(j)} e_s^{(j)} \rangle e_{s'}^{(j')} \\
&= \sum_{j', s'} \delta_{j'j} U_{s's}^{(j)} e_{s'}^{(j')} \\
&= \sum_{s'} U_{s's}^{(j)} e_{s'}^{(j)}.
\end{aligned}$$

Thus, we get

$$\left\langle e_o^{(k)}, \tau_l(e_r^{(l)})e_s^{(j)}\right\rangle = \sum_{r',s',o'} \overline{U_{o'o}^{(k)}}\, U_{r'r}^{(l)}\, U_{s's}^{(j)} \left\langle e_{o'}^{(k)}, \tau_l(e_{r'}^{(l)})e_{s'}^{(j)}\right\rangle.$$

The left-hand side does not depend on the group element, so we can write

$$\left\langle e_o^{(k)}, \tau_l(e_r^{(l)})e_s^{(j)}\right\rangle = \frac{1}{|G|}\sum_{a\in G}\left\langle e_o^{(k)}, \tau_l(e_r^{(l)})e_s^{(j)}\right\rangle,$$

then by using equation (3), one has

$$\begin{aligned}\left\langle e_o^{(k)}, \tau_l(e_r^{(l)})e_s^{(j)}\right\rangle &= \frac{1}{|G|}\sum_{r',s',o'}\sum_{a\in G} \overline{U_{o'o}^{(k)}(a)}\, U_{r'r}^{(l)}(a)\, U_{s's}^{(j)}(a) \left\langle e_{o'}^{(k)}, \tau_l(e_{r'}^{(l)})e_{s'}^{(j)}\right\rangle \\ &= \frac{1}{d_k}\sum_{r',s',o'} \langle ko|lr, js\rangle\langle lr', js'|ko'\rangle \left\langle e_{o'}^{(k)}, \tau_l(e_{r'}^{(l)})e_{s'}^{(j)}\right\rangle.\end{aligned}$$

Equivalently,

$$\left\langle e_o^{(k)}, \tau_l(e_r^{(l)})e_s^{(j)}\right\rangle = \langle ko|lr, js\rangle\, T(k,j,l), \tag{4}$$

where

$$T(k,j,l) = \frac{1}{d_k}\sum_{r',s',o'} \langle lr', js'|ko'\rangle \left\langle e_{o'}^{(k)}, \tau_l(e_{r'}^{(l)})e_{s'}^{(j)}\right\rangle.$$

The significant of this result is that we can split the inner product into the Clebsch-Gordon coefficient and a term depending on k, j, l alone (and not on the basis of the invariant subspace). In Dirac notation,

$$\left\langle e_o^{(k)}, \tau_l(e_r^{(l)})e_s^{(j)}\right\rangle := \langle ko|\tau_r^{(l)}|js\rangle \quad , \quad T(k,j,l) := \langle k\|\tau^{(l)}\|j\rangle.$$

Often, $\langle k\|\tau^{(l)}\|j\rangle$ is called the reduced matrix element. Equation (5) then reads

$$\langle ko|\tau_r^{(l)}|js\rangle = \langle ko|lr, js\rangle\langle k\|\tau^{(l)}\|j\rangle,$$

where

$$\langle k\|\tau^{(l)}\|j\rangle = \frac{1}{d_k}\sum_{r',s',o'} \langle lr', js'|ko'\rangle\langle ko'|\tau_{r'}^{(l)}|js'\rangle.$$

4 Reference

[1] Groups and their representations, lecture notes by S. Richard.
[2] Proof of the selection rule, report by Y. Li.
[3] Theorie des groupes pour la physique, lecture notes by W. Amrein.