Heisenberg Group

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1 Introduction

This report aims to introduce the Heisenberg group H named after Werner Heisenberg and solves relevant exercises.

2 Definition

The Heisenberg group H is the group of 3×3 upper triangular matrices A of the form

$$A = \begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

where a_1, a_2 and a_3 are arbitrary real numbers. It is clearly associative as it is a group of matrices.

Let B also take the form

$$B = \begin{pmatrix} 1 & b_1 & b_2 \\ 0 & 1 & b_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

where b_i $(i = 1, 2, 3) \in \mathbb{R}$. And AB is written as

$$AB = \begin{pmatrix} 1 & a_1 + b_1 & b_2 + a_1 b_3 + a_2 \\ 0 & 1 & a_3 + b_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

It is easy to check that the product of two matrices AB is again of the form (1). and it is clear that the identity matrix is of the form (1) by substituting zero to a_1, a_2, a_3.

Furthermore, a direct computation shows that if A is as in (1), then

$$A^{-1} = \begin{pmatrix} 1 & -a_1 & a_1 a_3 - a_2 \\ 0 & 1 & -a_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{4}$$

It can be checked that $AA^{-1} = A^{-1}A = I$.

Therefore, H is a subgroup of $\mathrm{GL}(3;\mathbb{R})$. In fact, H is also a matrix Lie group.

3 Exercises

> Exercise 1
>
> Determine the center $Z(H)$ of the Heisenberg group H. Show that the quotient group $H/Z(H)$ is abelian.

From Definition 1.2.11 in the lecture notes, the center of a Lie group H is defined by

$$Z(H) = \{A \in H \mid AB = BA \text{ for all } B \in H\} \tag{5}$$

Using eq.(3), $AB = BA$ is written as

$$\begin{pmatrix} 1 & a_1 + b_1 & b_2 + a_1 b_3 + a_2 \\ 0 & 1 & a_3 + b_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 + b_1 & b_2 + a_3 b_1 + a_2 \\ 0 & 1 & a_3 + b_3 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

From this, one can obtain

$$a_1 b_3 = a_3 b_1 \tag{7}$$

this equation is satisfied for any real numbers b_1, b_2, b_3 only when a_1 and a_3 are zero. Therefore $Z(H)$ can be given by

$$Z(H) = \left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\} \tag{8}$$

From now on, let us show that the quotient group $H/Z(H)$ is abelian. This group is defined by

$$H/Z(H) = \{[A] = AZ(H) = Z(H)A \mid A \in H\} \tag{9}$$

where $Z(H)$ is an abelian and normal subgroup of H from the exercise 1.2.12. It is sufficient to consider the left coset. $H/Z(H)$ is abelian if and only if $[A][B] = [B][A]$ for any $A, B \in H$. By definition, one has

$$\begin{aligned} [A][B] &= [AB] \\ &= ABZ(H) \\ &= \left\{ AB \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b_1 & b_2 \\ 0 & 1 & b_3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} 1 & a_1 + b_1 & a + (a_2 + b_2) + a_1 b_3 \\ 0 & 1 & a_3 + b_3 \\ 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} 1 & a_1 + b_1 & c \\ 0 & 1 & a_3 + b_3 \\ 0 & 0 & 1 \end{pmatrix} \mid c \in \mathbb{R} \right\} \end{aligned} \tag{10}$$

Likewise,

$$\begin{aligned}[B][A] &= [BA] \\ &= \left\{ \begin{pmatrix} 1 & a_1+b_1 & a+(a_2+b_2)+b_1a_3 \\ 0 & 1 & a_3+b_3 \\ 0 & 0 & 1 \end{pmatrix} \mid a \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} 1 & a_1+b_1 & c \\ 0 & 1 & a_3+b_3 \\ 0 & 0 & 1 \end{pmatrix} \mid c \in \mathbb{R} \right\} \end{aligned} \tag{11}$$

As shown above, it is possible to use $c \in \mathbb{R}$ as (1,3) element since a can take any real number. It is confirmed from equation (10) and (11) that $[A][B] = [B][A]$ and thus $H/Z(H)$ is abelian.

Exercise 2

Show that the exponential mapping from the Lie algebra of the Heisenberg group to the Heisenberg group is one-to-one and onto.

By definition, one has for $t \in \mathbb{R}$

$$e^{tX} = \sum_{j=0}^{\infty} \frac{(tX)^j}{j!} \tag{12}$$

Let us compute e^{tX}, where

$$tX = \begin{pmatrix} 0 & tx & ty \\ 0 & 0 & tz \\ 0 & 0 & 0 \end{pmatrix} \tag{13}$$

with any $x, y, z \in \mathbb{R}$. Note that

$$(tX)^2 = \begin{pmatrix} 0 & 0 & t^2xz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and that

$$(tX)^3 = 0$$

Therefore,

$$\begin{aligned} e^{tX} &= \frac{(tX)^0}{0!} + \frac{(tX)^1}{1!} + \frac{(tX)^2}{2!} + \sum_{j=3}^{\infty} \frac{(tX)^j}{j!} \\ &= \begin{pmatrix} 1 & tx & ty+t^2xz/2 \\ 0 & 1 & tz \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{14}$$

By definition, the Lie algebra of H denoted $L(H)$ is the set of all matrices Y such that e^{tY} is in H for all real numbers t.

$$L(H) = \{Y \mid e^{tY} \in H \ \text{ for all } \ t \in \mathbb{R}\} \tag{15}$$

Thus the matrix X is in $L(H)$ as $e^{tX} \in H$ for any $t \in \mathbb{R}$.

Let us now consider the exponential mapping from some element $X \in L(H)$ to $e^X \in H$.

Surjective proof:

The component x, z in X (eq.(13)) can take any real number, so the (1,2) and (2,3) elements in e^X (eq.(14)) can trivially be any real number. It is also clear that the element $y + xz/2$ can be any real number as y can take any real number for any fixed x, z. Therefore this mapping is surjective.

Injective proof:

Let X' denote X replacing x, y, z with x', y', z' and suppose $e^X = e^{X'}$, that is

$$\begin{aligned} x &= x' \\ y + xz/2 &= y' + x'z'/2 \\ z &= z' \end{aligned}$$

One can also derive $y = y'$ as it is clear that $xz/2 = x'z'/2$ from $x = x',\ z = z'$. It is finally confirmed that $X = X'$, which satisfies the condition for injective.

From the discussion above, it is proved that the exponential mapping from $X \in L(H)$ to $e^X \in H$ is bijective.

4 Application in physics

The following three elements form a basis for the Lie algebra of the Heisenberg group $L(H)$

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{16}$$

It is known that these basis elements satisfy the commutation relations,

$$[X, Y] = Z;\ [X, Z] = 0;\ [Y, Z] = 0 \tag{17}$$

The name "Heisenberg group" is motivated by the preceding relations, which have the same form as the canonical commutation relations in quantum mechanics,

$$[\hat{x}, \hat{p}] = i\hbar I;\ [\hat{x}, i\hbar I] = 0;\ [\hat{p}, i\hbar I] = 0 \tag{18}$$

where $\hat{x}$ is the position operator, $\hat{p}$ is the momentum operator, and $\hbar$ is Planck's constant.

References

[1] Lie Groups, Lie Algebras, and Representations An Elementary Introduction, Brian C.Hall

[2] Groups and their representations, Serge Richard