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Group theory - SML

Exercise 2.48. Check that the regular representation is indeed a representation. Is it a unitary representation?

Answer:

+, First we check if the regular representation is indeed a representation.

we get the regular representation of G (a finite group) is given by $(\mathcal{H}^{reg}, U^{reg})$ with $[U^{reg}(a)f](b) = f(a^{-1}b)$ $\forall f \in \mathcal{H}^{reg} := \ell^2(G)$

We need to check 2 conditions:

- $U^{reg}(e) = \mathbb{1}$

we have $[U^{reg}(e)f](b) = [\mathbb{1}f](b) = f(b) = f(e^{-1}b)$

$\rightarrow$ satisfied

- $U^{reg}(xy) = U^{reg}(x)\, U^{reg}(y)$ $\forall x, y \in G$

we have

$[U^{reg}(xy)f](b) = f((xy)^{-1}b) = f(y^{-1}x^{-1}b) = [U^{reg}(y)f](x^{-1}b)$

$\Leftrightarrow [U^{reg}(xy)f](b) = [U^{reg}(x)\, U^{reg}(y) f](b)$

$\rightarrow$ Q.E.D

So regular representation is indeed a representation.

+, Now we check if regular representation is a unitary representation.

A representation is a unitary representation of group G if a linear representation U of G on $\mathcal{H}$ s.t $U(g)$ is a unitary operator $\forall g \in G$.

We check if $U^{reg}(g)$ is a unitary operator $\forall g \in G$.

Because $f, g \in \mathcal{H}^{reg} := \ell^2(G)$. We get

$\langle f, g \rangle = \sum_{b \in G} f(b) \overline{g(b)}$ ①

We consider $\langle U^{reg}(a) f, U^{reg}(a) g \rangle$. We get

$\langle U^{reg}(a) f, U^{reg}(a) g \rangle = \sum_{b \in G} U^{reg}(a) f(b)\, \overline{U^{reg}(a) g(b)}$

$= \sum_{b \in G} f(a^{-1}b)\, \overline{g(a^{-1}b)}$

We set $c = a^{-1}b$, we get

$\langle U^{reg}(a) f, U^{reg}(a) g \rangle = \sum_{c \in G} f(c)\, \overline{g(c)}$ ② $(c = a^{-1}b \in G)$

From ① and ② $\rightarrow \langle U^{reg}(a) f, U^{reg}(a) g \rangle = \langle f, g \rangle$

From there, we can see that $U^{reg}(a)$ preserve inner products.
Moreover, we can see that:
$U^{reg}(a): H \to H$ and $U^{reg}(a)$ is invertible $\left(U^{reg}(a^{-1}) = U^{reg}(a)^{-1}\right)$
$\Rightarrow U^{reg}(a)$ is unitary operator
$\to U^{reg}$ is unitary representation.
$\to$ Q.E.D