Baker-Campbell-Hausdorff Theorem

NGUYEN Thanh, PENG Qi

January 28, 2023

1 Introduction

As we have mentioned during the lecture, the exponential operation on group elements of Lie groups is different from which for scalars (i.e., $e^Z = e^{X+Y} = e^X e^Y$), while according to the Baker-Campbell-Hausdorff Theorem the solution for a Lie algebraic term is going to

$$e^Z = e^X e^Y,$$

for $X, Y \in L(G)$, and we shall give the equation for the solution shown as

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots \tag{1}$$

where "$\dots$" indicates some terms involving high commutators of X, Y. In this report, we would like to give proof by introducing several new theorems, lemmas, and propositions of Lie groups and Lie algebras extended from the lecture, which enables us to give the final proof to the general Baker-Campbell-Hausdorff Theorem, shown in (1).

2 Proof

2.1 Lie group and Lie algebra

As we have seen in the lecture, we can go from the Lie algebra to the Lie group (at least part of it) by exponential mapping. Now, we want something in the opposite direction and this is called logarithmic mapping. Since we will mainly deal with invertible matrices, let us define the logarithm of an $n \times n$ matrix A by

$$\log(A) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(A - I)^m}{m}$$

whenever it is convergent, where I is the identity matrix. This definition is analog to the definition of the logarithm of a complex number z that converges when $|z - 1| < 1$. Similarly, the series for matrices is convergent if $\|A - I\| < 1$ and we will assume that this is always the case, i.e., we restrict our attention to the neighborhood of the identity element, whenever we deal with the logarithmic mapping.

Let us consider some important relations between the Lie group and its Lie algebra. If C is an invertible $n \times n$ matrix and X is some arbitrary $n \times n$ matrix, one has

$$\begin{aligned} e^{CXC^{-1}} &= \sum_{m=0}^{\infty} \frac{1}{m!}(CXC^{-1})^m \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} CX^m C^{-1} \\ &= C\left(\sum_{m=0}^{\infty} \frac{X^m}{m!}\right) C^{-1} \\ &= Ce^X C^{-1}. \end{aligned}$$

Then, if we let $t \in \mathbb{R}$, C be an element of a linear Lie group G, and X be an element of the corresponding Lie algebra $L(G)$, we find that

$$e^{t(CXC^{-1})} = e^{C(tX)C^{-1}} = Ce^{tX}C^{-1}.$$

Since $C, e^{tX}, C^{-1} \in G$, we have $e^{t(CXC^{-1})} \in G$ so $CXC^{-1} \in L(G)$.

Theorem: Let G, H be linear Lie groups and $L(G)$, $L(H)$ be their corresponding Lie algebras. If $\Phi : G \to H$ is a (continuous) group homomorphism, there exists a unique real linear map $\phi : L(G) \to L(H)$ such that for all $X \in L(G)$, one has

$$\Phi(e^X) = e^{\phi(X)}.$$

Also, the map ϕ has the following properties:

1. $\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1} \quad \forall X \in L(G), A \in G,$
2. $\phi([X, Y]) = [\phi(X), \phi(Y)] \quad \forall X, Y \in L(G),$
3. $\phi(X) = \frac{d}{dt}\Phi(e^{tX})\big|_{t=0} \quad \forall X \in L(G).$

Readers can refer to p.45-47, [H], for the detail of the proof. The argument of ϕ in property 1 is well-defined by applying the previous result. Using the linearity of ϕ and property 2, we can see that ϕ preserves the law for the Lie algebra and it is, hence, a Lie algebra homomorphism. However, the converse is not true in general as it requires G to be simply connected (see p.76-80, [H]). The relations between these Lie groups and Lie algebras can be summarized by the following diagram

$$\begin{tikzcd} G \arrow[rr, "\Phi"] \arrow[d, shift left, "\text{``log''}"] & & H \arrow[d, shift left, "\text{``log''}"] \\ \mathrm{L}(G) \arrow[rr, "\phi"'] \arrow[u, shift left, "\text{``exp''}"] & & \mathrm{L}(H) \arrow[u, shift left, "\text{``exp''}"] \end{tikzcd}$$

when the exponential and the logarithmic functions are well-defined.

2.2 Adjoint mapping

Next, we will define adjoint mapping. For $A \in G$, we can define the linear map

$$\begin{aligned} \mathrm{Ad}_A : \ L(G) &\to L(G), \\ X \ &\mapsto AXA^{-1}. \end{aligned}$$

Since $L(G)$ can be considered as a vector space, we can also regard the map

$$\begin{aligned} \mathrm{Ad} : \ G &\to \mathscr{L}(L(G)), \\ A &\mapsto \mathrm{Ad}_A. \end{aligned}$$

Now, we will show that $H := \{\mathrm{Ad}_A\}_{A \in G}$, in fact, forms a group. This means that H needs to have the following properties

1. Associativity:
   For any $A, B, C \in G$ and $X \in L(G)$,

$$\begin{aligned} (\mathrm{Ad}_A\mathrm{Ad}_B)\mathrm{Ad}_C(X) &= \mathrm{Ad}_A\mathrm{Ad}_B(CXC^{-1}) \\ &= \mathrm{Ad}_A(BCXC^{-1}B^{-1}) \\ &= ABCXC^{-1}B^{-1}A^{-1}, \end{aligned}$$

$$
\begin{aligned}
\mathrm{Ad}_A(\mathrm{Ad}_B\mathrm{Ad}_C)(X) &= A\left(\mathrm{Ad}_B\mathrm{Ad}_C(X)\right)A^{-1} \\
&= A\left(\mathrm{Ad}_B(CXC^{-1})\right)A^{-1} \\
&= ABCXC^{-1}B^{-1}A^{-1},
\end{aligned}
$$

so $(\mathrm{Ad}_A\mathrm{Ad}_B)\mathrm{Ad}_C = \mathrm{Ad}_A(\mathrm{Ad}_B\mathrm{Ad}_C)$.

2. Existence of an identity element:
   Let E be the identity element of G. Then for any $A \in G$, $X \in L(G)$,

$$
\begin{cases}
\mathrm{Ad}_A\mathrm{Ad}_E(X) = \mathrm{Ad}_A(EXE^{-1}) = \mathrm{Ad}_A(X) \\
\mathrm{Ad}_E\mathrm{Ad}_A(X) = E\left(\mathrm{Ad}_A(X)\right)E^{-1} = \mathrm{Ad}_A(X)
\end{cases},
$$

   so $\mathrm{Ad}_A\mathrm{Ad}_E = \mathrm{Ad}_E\mathrm{Ad}_A = \mathrm{Ad}_A$, meaning that Ad_E is the identity element of H.

3. Existence of an inverse:
   For any $A \in G$,

$$
\begin{cases}
\mathrm{Ad}_A\mathrm{Ad}_{A^{-1}}(X) = \mathrm{Ad}_A(A^{-1}XA) = AA^{-1}XAA^{-1} = X = \mathrm{Ad}_E(X) \\
\mathrm{Ad}_{A^{-1}}\mathrm{Ad}_A(X) = \mathrm{Ad}_{A^{-1}}(AXA^{-1}) = A^{-1}AXA^{-1}A = X = \mathrm{Ad}_E(X)
\end{cases},
$$

   so $\mathrm{Ad}_A\mathrm{Ad}_{A^{-1}} = \mathrm{Ad}_{A^{-1}}\mathrm{Ad}_A = \mathrm{Ad}_E$, meaning that $\mathrm{Ad}_{A^{-1}} = \mathrm{Ad}_A^{-1}$.

□

Thus, Ad is a map between groups. Let us also show that Ad is a homomorphism as well. For any $A, B \in G$, one has

$$
\begin{aligned}
\mathrm{Ad}_A\mathrm{Ad}_B(X) &= \mathrm{Ad}_A(BXB^{-1}) \\
&= ABXB^{-1}A^{-1} \\
&= (AB)X(AB)^{-1} \\
&= \mathrm{Ad}_{AB}(X),
\end{aligned}
$$

so $\mathrm{Ad}_A\mathrm{Ad}_B = \mathrm{Ad}_{AB}$. This means that Ad preserves the group law and it is, therefore, a homomorphism. □

We then apply the theorem in section 2.1 to the map $\Phi = \mathrm{Ad}$ with H defined as above. Denote the corresponding Lie algebra homomorphism by $\phi = \mathrm{ad}$, the Lie algebra of H is given by

$$
L(H) = \{\mathrm{ad}_X | X \in L(G)\}.
$$

where we have

$$
\mathrm{ad}_X = \frac{d}{dt}\mathrm{Ad}(e^{tX})\Big|_{t=0}
$$

Previously, we define Ad_A by its action on $L(G)$, so we can also define ad_X by its action on $L(G)$. More precisely, for any $X, Y \in L(G)$,

$$
\begin{aligned}
\mathrm{ad}_X(Y) &= \frac{d}{dt}\mathrm{Ad}(e^{tX})(Y)\Big|_{t=0} \\
&= \frac{d}{dt}e^{tX}Ye^{-tX}\Big|_{t=0} \\
&= \left[X(e^{tX}Ye^{-tX}) - (e^{tX}Ye^{-tX})X\right]\Big|_{t=0} \\
&= XY - YX = [X, Y].
\end{aligned}
$$

This is consistent with the definition from the lecture.

Proposition: For any $X, Y \in L(G)$, we have

$$
e^{\mathrm{ad}_X}(Y) = \mathrm{Ad}_{e^X}(Y) = e^XYe^{-X}. \tag{2}
$$

This can be derived easily by using the definition of Ad and the theorem from before, but we can also get this by direct calculation as follows. First, we will prove by induction that

$$(\mathrm{ad}_X)^m(Y) = \sum_{k=0}^{m} \binom{m}{k} X^k Y (-X)^{m-k}, \tag{3}$$

where

$$(\mathrm{ad}_X)^m(Y) = [X, \cdots [X, [X, Y]] \cdots].$$

It is easy to show that for $m = 0$ the above quality holds. We now assume the equation holds for some $m > 0$, then consider $m + 1$ case

$$\begin{aligned}
(\mathrm{ad}_X)^{m+1}(Y) &= [X, (\mathrm{ad}_X)^m(Y)] \\
&= \sum_{k=0}^{m} \binom{m}{k} X^{k+1} Y (-X)^{m-k} + \sum_{k=0}^{m} \binom{m}{k} X^k Y (-X)^{m-k+1} \\
&= \sum_{k=1}^{m+1} \binom{m}{k-1} X^k Y (-X)^{m-k+1} + \sum_{k=0}^{m} \binom{m}{k} X^k Y (-X)^{m-k+1} \\
&= \sum_{k=1}^{m} \left[\binom{m}{k-1} + \binom{m}{k} \right] X^k Y (-X)^{m+1-k} + X^{m+1} Y + Y (-X)^{m+1}. \qquad (\star)
\end{aligned}$$

We know that from Pascal's triangle,

$$\begin{aligned}
\binom{m}{k-1} + \binom{m}{k} &= \frac{m!}{(k-1)!(m+1-k)!} + \frac{m!}{k!(m-k)!} \\
&= \frac{m!}{k!(m+1-k)!} [k + (m+1-k)] \\
&= \binom{m+1}{k}.
\end{aligned}$$

Therefore, $(\star)$ gives

$$\begin{aligned}
(\mathrm{ad}_X)^{m+1}(Y) &= \sum_{k=1}^{m} \binom{m+1}{k} X^k Y (-X)^{m+1-k} + X^{m+1} Y + Y (-X)^{m+1} \\
&= \sum_{k=0}^{m+1} \binom{m+1}{k} X^k Y (-X)^{m+1-k},
\end{aligned}$$

so we have proved equation (3). Now, for the left-hand side of (2), let us Taylor expand the exponential

$$\begin{aligned}
e^{\mathrm{ad}_X}(Y) &= \sum_{m=0}^{\infty} \frac{(\mathrm{ad}_X)^m}{m!}(Y) \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{1}{m!} \binom{m}{k} X^k Y (-X)^{m-k} \\
&= \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{X^k Y (-X)^{m-k}}{k!(m-k)!} \\
&= \sum_{m,n=0}^{\infty} \frac{X^m Y (-X)^n}{m! n!}
\end{aligned}$$

where we have used the Cauchy product at the final step. Also, if we consider the two equations on the right-hand side of (2)

$$
\begin{aligned}
\mathrm{Ad}_{e^X}(Y) = e^X Y e^{-X} &= \left(\sum_{m=0}^{\infty} \frac{X^m}{m!}\right) Y \left(\sum_{n=0}^{\infty} \frac{(-X)^n}{n!}\right) \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left(\frac{X^m}{m!} Y \frac{(-X)^n}{n!}\right) \\
&= \sum_{m,n=0}^{\infty} \frac{X^m Y (-X)^n}{m!n!} \\
&= e^{\mathrm{ad}_X}(Y)
\end{aligned}
$$

Thus we proved the proposition as shown in equation (2). □

2.3 Derivative of the exponential mapping

In this section, we would like to introduce the exponential mapping derivative briefly. Let $X, Y \in L(G)$, then the derivative

$$
\frac{d}{dt} e^{X+tY}\Big|_{t=0} = e^X \left\{\frac{I - e^{-\mathrm{ad}_{Z(t)}}}{\mathrm{ad}_{Z(t)}}\right\}(Y) = e^X \left\{Y - \frac{[X,Y]}{2!} + \frac{[X,[X,Y]]}{3!} - \cdots\right\}, \tag{4}
$$

This is the special case when we set $Z(t) = X + tY$. More generally, we have

$$
\frac{d}{dt} e^{Z(t)} = e^{Z(t)} \left\{\frac{I - e^{\mathrm{ad}_{Z(t)}}}{\mathrm{ad}_{Z(t)}}\right\} \frac{dZ(t)}{dt}. \tag{5}
$$

The proof of (4) and (5) is given on p.70-73, [H].

Now, we shall proceed with the proof of the "ultimate" theorem.

2.4 Proof of the Baker-Campbell-Hausdorff Theorem

Following the definition of logarithm mapping, let us define the function $Z(t)$ where $t \in \mathbb{R}$ by

$$
Z(t) = \log(e^X e^{tY}). \tag{6}
$$

This means that

$$
e^{Z(t)} = e^X e^{tY}. \tag{7}
$$

Using equation (7), we find that

$$
e^{-Z(t)} \frac{d}{dt} e^{Z(t)} = (e^X e^{tY})^{-1} e^X e^{tY} Y = Y.
$$

Focusing on the left-hand side of (7), we can use the expression for the derivative of an exponential from the previous section

$$
e^{-Z(t)} \frac{d}{dt} e^{Z(t)} = \left\{\frac{I - e^{-\mathrm{ad}_{Z(t)}}}{\mathrm{ad}_{Z(t)}}\right\} \left(\frac{dZ(t)}{dt}\right).
$$

Combining the two equations together gives

$$
\left\{\frac{I - e^{-\mathrm{ad}_{Z(t)}}}{\mathrm{ad}_{Z(t)}}\right\} \left(\frac{dZ(t)}{dt}\right) = Y
$$

$$
\text{or} \quad \frac{dZ(t)}{dt} = \left\{\frac{I - e^{-\mathrm{ad}_{Z(t)}}}{\mathrm{ad}_{Z(t)}}\right\}^{-1} (Y). \tag{8}
$$

By the proposition in section 2.2, we have

$$e^{\mathrm{ad}_X}(Y) = \mathrm{Ad}_{e^X} Y = e^X Y e^{-X}.$$

Recall that $e^{Z(t)} = e^X e^{tY}$. Using the fact that Ad is a Lie group homomorphism, one has

$$\mathrm{Ad}_{e^{Z(t)}} = \mathrm{Ad}_{e^X} \mathrm{Ad}_{e^{tY}},$$

so the proposition gives

$$e^{\mathrm{ad}_{Z(t)}} = e^{\mathrm{ad}_X} e^{t\mathrm{ad}_Y}.$$

By taking the logarithm of the above equation, we get

$$\mathrm{ad}_{Z(t)} = log(e^{\mathrm{ad}_X} e^{t\mathrm{ad}_Y}).$$

Plugging this into equation (8),

$$\frac{dZ(t)}{dt} = \left\{ \frac{I - (e^{\mathrm{ad}_X} e^{t\mathrm{ad}_Y})^{-1}}{log(e^{\mathrm{ad}_X} e^{t\mathrm{ad}_Y})} \right\}^{-1} (Y). \tag{9}$$

To determine the expression in $\{...\}^{-1}$, we consider the analogous function on the complex number z

$$g(z) = \left\{ \frac{1 - 1/z}{\log(z)} \right\}^{-1} = \frac{\log(z)}{1 - 1/z}. \tag{10}$$

This equation is defined and analytic in the disk $|z - 1| < 1$. Then, letting $z \to e^{\mathrm{ad}_X} e^{t\mathrm{ad}_Y}$, we can write (9) as

$$\frac{dZ(t)}{dt} = g(e^{\mathrm{ad}_X} e^{t\mathrm{ad}_Y})(Y),$$

By doing an integration from $t = 0$ to $t = 1$ and using the definition given in (6), we finally get

$$Z(1) = \log(e^X e^Y) = X + \int_0^1 g(e^{\mathrm{ad}_X} e^{t\mathrm{ad}_Y})(Y) dt. \tag{11}$$

This is the general form of the Baker-Campbell-Hausdorff Theorem.

However, this is not a very useful expression due to the integral, so let us try to calculate it more explicitly. Starting from expanding equation (10) in power series of $(z - 1)$

$$\begin{aligned}
g(z) &= \frac{\log(z)}{1 - 1/z} = \frac{z\log(z)}{z - 1} \\
&= \frac{[1 + (z - 1)][(z - 1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots]}{(z - 1)} \\
&= [1 + (z - 1)][1 - \frac{(z - 1)}{2} + \frac{(z - 1)^2}{3} - \cdots] \\
&= 1 + \frac{1}{2}(z - 1) - \frac{1}{6}(z - 1)^2 + \cdots,
\end{aligned}$$

then the close-form expression for g is

$$g(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m(m + 1)} (z - 1)^m \tag{12}$$

Since we make the substitution of $z \to e^{\mathrm{ad}_X} e^{t\mathrm{ad}_Y}$, we will first calculate

$$\begin{aligned}
e^{\mathrm{ad}_X} e^{t\mathrm{ad}_Y} - I &= \left[I + \mathrm{ad}_X + \frac{(\mathrm{ad}_X)^2}{2} + \cdots \right] \left[I + t\mathrm{ad}_X + \frac{t^2(\mathrm{ad}_Y)^2}{2} + \cdots \right] - I \\
&= \mathrm{ad}_X + t\mathrm{ad}_Y + t\mathrm{ad}_X \mathrm{ad}_Y + \frac{(\mathrm{ad}_X)^2}{2} + \frac{t^2(\mathrm{ad}_Y)^2}{2} + \cdots - I.
\end{aligned}$$

Then from (12), we have

$$\begin{aligned}
g(e^{\mathrm{ad}_X} e^{t\mathrm{ad}_Y}) &= I + \frac{1}{2}[\mathrm{ad}_X + t\mathrm{ad}_Y + t\mathrm{ad}_X\mathrm{ad}_Y + \frac{(\mathrm{ad}_X)^2}{2} + \frac{t^2(\mathrm{ad}_Y)^2}{2} + \cdots] \\
&\quad - \frac{1}{6}[\mathrm{ad}_X + t\mathrm{ad}_Y]^2 + \cdots \\
&= I + \frac{1}{2}\mathrm{ad}_X + \frac{1}{2}t\mathrm{ad}_Y + \frac{1}{2}t\mathrm{ad}_X\mathrm{ad}_Y + \frac{(\mathrm{ad}_X)^2}{4} + \frac{t^2(\mathrm{ad}_Y)^2}{4} \\
&\quad - \frac{1}{6}[(\mathrm{ad}_X)^2 + t^2(\mathrm{ad}_Y)^2 + t\mathrm{ad}_X\mathrm{ad}_Y + t\mathrm{ad}_Y\mathrm{ad}_X] + \cdots,
\end{aligned}$$

where we have neglected higher-order terms. Now, substitute the above result into the integration in (11) and use the fact that $\mathrm{ad}_X(Y) = [X,Y]$ and $\mathrm{ad}_Y(Y) = [Y,Y] = 0$ to get

$$\begin{aligned}
\log(e^X e^Y) &= X + \int_0^1 \left(Y + \frac{1}{2}[X,Y] + \frac{1}{4}[X,[X,Y]] - \frac{1}{6}[X,[X,Y]] - \frac{t}{6}[Y,[X,Y]] + \cdots \right) dt \\
&= X + Y + \frac{1}{2}[X,Y] + \left(\frac{1}{4} - \frac{1}{6}\right)[X,[X,Y]] - \frac{1}{6}[Y,[X,Y]] \int_0^1 t\,dt + \cdots \\
&= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots,
\end{aligned}$$

which is the same as (1), the Baker-Campbell-Hausdorff Theorem in the form we have seen during the lecture. □