On the second countability of $\mathbb{R}$ and $\mathbb{R}^n$

Magnus B. Lyngby (612206006)

January 24, 2023

1 Introduction

In this report I first give a proof that $\mathbb{R}$ is second countable, by constructing an arbitrary open set. Then I generalize this method to construct an arbitrary open box in $\mathbb{R}^n$. Finally, I give a simple proof that $\mathbb{R}^n$ is second countable.

2 Main Proof

I want to show that there exists a countable basis for $\mathbb{R}$. Let the open set

$$B(r, x_0) := \{x \in \mathbb{R} \mid |x - x_0| < r \text{ where } r, x_0 \in \mathbb{Q}\}. \tag{1}$$

Since these open sets are only defined for rational values of r, r_0 they are countable. It is clear that the open intervals (a,b), $a, b \in \mathbb{R}$ form an uncountable basis on $\mathbb{R}$. Therefore, given one of these intervals I wish to show it can be covered by an arbitrary union of open sets of the form (1). Assume (a, b) is an open interval where $a, b \notin Q$, since otherwise the exercise is trivial. There exists some $\mathbb{Q} \ni x_0 \in (a, b)$. Either this point is closest to a or b, or lies exactly in the middle of the interval. Let it be closest to a. In that case I define $a_1 := \frac{x_0+a}{2}$ which gives me a new open interval (a, a_1). Find some $\mathbb{Q} \ni x_{1a} \in (a, a_1)$ and define $B(|x_{1a} - x_0|, x_0)$. This set is inside of the interval, and crucially covers at least half of the distance from x_0 to a.

Continue, defining $a_2 := \frac{a_1+a}{2}$ and $\mathbb{Q} \ni x_{2a} \in (a, a_2)$ as well as $B(|x_{2a} - x_{1a}|, x_{1a})$. Continuing this construction, we get a set of intervals:

$$B(|x_{1a} - x_0|, x_0), B(|x_{2a} - x_{1a}|, x_{1a}), B(|x_{3a} - x_{2a}|, x_{2a}), \ldots \tag{2}$$

that cover

$$[a_1, x_0], [a_2, a_1], [a_3, a_2], \ldots \tag{3}$$

Since $a_n := \frac{a_{n-1}+a}{2}$ it is clear $a_n \to a$ as $n \to \infty$ such that in the limit the intervals cover $(a, x_0]$. To cover the rest, let $\mathbb{Q} \ni y_0 \in (b - (x_0 - a), b)$. This will be an open set in all cases unless x_0 is the middle point, in which case I just define $y_0 = x_0$. Now repeat the above construction, but for intervals going from y_0 to b, covering $[y_0, b)$. At last define $B(|\frac{x_0-y_0}{2}|, \frac{x_0+y_0}{2})$ which covers (x_0, y_0). Together, these three constructions cover $(a, x_0] \cup (x_0, y_0) \cup [y_0, b) = (a, b)$, which is what we required. Thus, $\mathbb{R}$ is second countable.

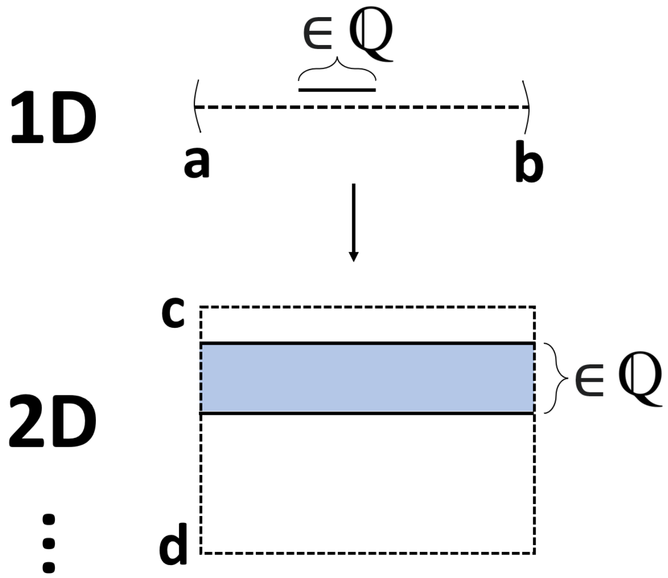

Figure 1: This figure shows the geometric construction of a hyperbox in $\mathbb{R}^n$ using only the union of a countable set of elements. In each step, the problem reduces to a 1 dimensional problem, which I already showed is possible to solve in the preceding section.

3 Sketch of generalization to $\mathbb{R}^n$

The method outlined above can be used to construct a hyperbox in $\mathbb{R}^n$. To see the construction geometrically I refer to Figure 1. First, let us be given a hyperbox:

$$V(a_1, b_1, \ldots, a_n, b_n) = \{(x_1, x_2, \ldots, x_N) = \boldsymbol{x} \in \mathbb{R}^n | a_k < x_k < b_k \ \forall k\} \tag{4}$$

Consider the set of line segments:

$$V_1(x_2, \ldots, x_n) := \{(x_1, x_2, \ldots, x_N) = \boldsymbol{x} \in \mathbb{R}^n | a_1 < x_1 < b_1, x_2, \ldots, x_n \text{ const.}\} \tag{5}$$

By defining[1]:

$$D_1(r, x_0) := \{(x_1, x_2, \ldots, x_N) = \boldsymbol{x} \in \mathbb{R}^n \mid |x_1 - x_0| < r \text{ where } r, x_0 \in \mathbb{Q}\}. \tag{6}$$

We can go through the exact construction as for the one-dimensional case, to produce a countable set of unions that produce any V_1. Let $\{V_1\}$ denote the set of all possible lines of the form given above. This set is uncountable, but we will consider a subset that is countable, in the following way. Consider the area and corresponding area segments:

$$V_2(x_3, \ldots, x_n) := \{\boldsymbol{x} \in \mathbb{R}^n | a_1 < x_1 < b_1, \ a_2 < x_2 < b_2, x_3, \ldots, x_n \text{ const.}\} \tag{7}$$
$$D_2(r, x_0) := \{L \in \{V_1\} \mid |x_2 - x_0| < r \text{ where } r, x_0 \in \mathbb{Q}\} \tag{8}$$

Where each L is implicitly a variable of x_2, and since $\mathbb{Q}$ is countable, $\{D_2(r, x_0)\}$ is countable. Given these two definitions, I can again go through the exact same construction as for the one-dimensional

[1] This definition is equivalent to (1).

case, to produce a countable set of unions that produce any V_2. Continuing this way n times gives me the set V, using only a countable set of elements. If none of this makes sense, hopefully the Figure does.

If the set of all V can be considered a basis on $\mathbb{R}^n$, then this also functions as a proof that $\mathbb{R}^n$ is second countable. However, there does exist a much simpler proof.

4 Problem 3.1.7: Proof that $\mathbb{R}^n$ is second countable

The above construction is very elaborate, and wholely unnecessary to prove that $\mathbb{R}^n$ is second countable. A much easier method is to say we are given a point $p \in \mathbb{R}^n$, and an arbitrary neighborhood V around p. Consider,

$$B(\boldsymbol{r}, \boldsymbol{r_0}) := \{\boldsymbol{x} \in \mathbb{R}^n \mid \ |\boldsymbol{x} - \boldsymbol{r_0}| < r\} \tag{9}$$

$$B_Q(\boldsymbol{r}, \boldsymbol{r_0}) := \{\boldsymbol{x} \in \mathbb{R}^n \mid \ |\boldsymbol{x} - \boldsymbol{r_0}| < r \text{ and } r, r_0 \in \mathbb{Q}\} \tag{10}$$

Since the neighborhood is an open set, we can construct around an arbitrary point p_0 a ball $B(\boldsymbol{r}, \boldsymbol{p_0}) \subset V$. Inside this ball, we can produce at a set of balls $\{B_{Q,0}\}$ with rational center and radius. Picking one of these balls that includes ${r_0}$[2] means we have constructed a ball with rational center and radius, which is inside the neighborhood V and contains p_0. Doing this for all the uncountable points $p_n \in V$ gives an uncountable union $\bigcup B_{Q,n}$ with each $B_{Q,n} \in \{B_Q\}$. By definition this union must be contained within V and contain all points of V, so it is clear that $\bigcup B_{Q,n} = V$. Therefore we have shown that $\{B_Q\}$ is a countable basis and thus that $\mathbb{R}^n$ is second countable.

[2] To directly prove that this is possible, one could imagine doing a construction like in Section 2 along a diameter of the ball, and therefore argue that since this construction covers the whole line at least one ball must cover p_0.