On Quotient Groups

Special Mathematics Lecture
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1 Introduction

In this report I will prove Proposition 1.2.9. [1], as well as show that $\left|\frac{G}{G_0}\right| = \frac{|G|}{|G_0|}$ for finite groups. The proof will be broken up into three sections, one for each sub-statement of Proposition 1.2.9.

2 Proof of Proposition 1.2.9

2.1 Proof that a necessary and sufficient condition for a normal subgroup is equality of left, right cosets

We want to prove:

$$G_0 \text{ is a normal subgroup of } G \iff {}_{G_0}[c] = [c]_{G_0} \ \forall c \in G \tag{1}$$

By using the definition of a normal subgroup:

$$G_0 = cG_0c^{-1} \ \forall c \in G \text{ Multiply both sides on the right by c} \tag{2}$$
$$\iff G_0c = cG_0 \ \forall c \in G \text{ Use the definition of left and right cosets} \tag{3}$$
$$\iff [c]_{G_0} = {}_{G_0}[c] \ \forall c \in G \tag{4}$$

By following the logic backwards we see that we have proved the statement.

2.2 Proof that the quotient group is well-defined

First I must prove that the product $[a]_{G_0}[b]_{G_0} := [ab]_{G_0}$ is well-defined for a normal subgroup. The product is well-defined if $\forall x \in [a]_{G_0}$ and $\forall y \in [b]_{G_0}$ we have $xy \in [ab]_{G_0}$. In words, the representative for the equivalence class does not matter. Assuming we arbitrarily pick x, y belonging to their respective classes we know:

$$[x]_{G_0} = [a]_{G_0} \tag{5}$$

$$[x]_{G_0} = [a]_{G_0} \tag{6}$$

This together with (3) lets us prove that the product is well-defined:

$$\begin{aligned} [xy]_{G_0} &= (G_0x)y = [x]_{G_0}y = [a]_{G_0}y = G_0ay = a(G_0y) \\ &= a[y]_{G_0} = a[b]_{G_0} = aG_0b = G_0ab = [ab]_{G_0} \end{aligned} \tag{7}$$

Which gives the desired result. Next we must show that the product defined on the equivalence classes forms a group. This is done by checking the three conditions in Def. 1.1.1.. Let $a, b, c \in G$:

$$([a]_{G_0}[b]_{G_0})[c]_{G_0} = [ab]_{G_0}[c]_{G_0} = [abc]_{G_0} \tag{8}$$

$$[a]_{G_0}([b]_{G_0}[c]_{G_0}) = [a]_{G_0}[bc]_{G_0} = [abc]_{G_0} \tag{9}$$

Thus the product is associative. Next observe that G_0 is an equivalence class, simply represented as $[e]_{G_0}$, and it acts as the identity:

$$[a]_{G_0}[e]_{G_0} = [ae]_{G_0} = [a]_{G_0} \tag{10}$$

$$[e]_{G_0}[a]_{G_0} = [ea]_{G_0} = [a]_{G_0} \tag{11}$$

Lastly I will show that $[a^{-1}]_{G_0}$ acts as an inverse to any equivalence class represented by $a \in G$:

$$[a]_{G_0}[a^{-1}]_{G_0} = [aa^{-1}]_{G_0} = [e]_{G_0} = G_0 \tag{12}$$

$$[a^{-1}]_{G_0}[a]_{G_0} = [a^{-1}a]_{G_0} = [e]_{G_0} = G_0 \tag{13}$$

All this together shows that the quotient group is well-defined, and is a group.

2.3 Showing the size of quotient groups for finite groups

I want to show $\left|\frac{G}{G_0}\right| = \frac{|G|}{|G_0|}$. We know the equivalence classes define a partition of G, per definition, and we know $|G_0|$ is the number of elements in the identity equivalence class. I want to show that every other equivalence class also has $|G_0|$ elements, since then the number of equivalence classes, $\left|\frac{G}{G_0}\right|$ would equal $\frac{|G|}{|G_0|}$, which is exactly what I want to show.

Recall that we can transform from $[e]_{G_0}$ to $[a]_{G_0}$ by a, and we can transform the other way by a^{-1}. Since the transformation is invertible, it must be bijective, which means every element in $[e]_{G_0}$ is mapped exactly once to every element in $[a]_{G_0}$. Since a was arbitrarily chosen, this means every equivalence class has size $|G_0|$, which proves the statement.

References

[1] Austen Vic and Richard Serge. *Groups and their representations*. 2022.