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① The Euclidean Group is a Lie Group.

② The Poincare Group is a Lie Group.

Consider matrices of the form $\text{aff}(M, a) = \begin{pmatrix} M & a \\ 0 & 1 \end{pmatrix}$ M denoting an $n \times n$ matrix with each element in \mathbb{R} a denoting an $n \times 1$ vector with each element in \mathbb{R} 0 denoting a $1 \times n$ row of 0 'sClaim: If M is invertible, $\text{aff}(M, a)$ forms a group under MultiplicationProofLet $\text{aff}(M_1, a_1) = A$, $\text{aff}(M_2, b_1) = B$ where M_1, M_2 are $n \times n$ matrices and a_1, b_1 are $n \times 1$ vectorsAdditionally, M_1, M_2 are invertible. Let X be the set of all matrices $\text{aff}(M, a)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & a'_1 \\ a_{21} & a_{22} & \dots & a_{2n} & a'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & a'_n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} & b'_1 \\ b_{21} & b_{22} & \dots & b_{2n} & b'_2 \\ \vdots & \vdots & & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nn} & b'_n \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

$$AB = \begin{pmatrix} \sum_{k=1}^n a_{1k} b_{k1} & \sum_{k=1}^n a_{1k} b_{k2} & \dots & \sum_{k=1}^n a_{1k} b_{kn} & a'_1 + \sum_{k=1}^n a_{1k} b'_k \\ \sum_{k=1}^n a_{2k} b_{k1} & \sum_{k=1}^n a_{2k} b_{k2} & \dots & \sum_{k=1}^n a_{2k} b_{kn} & a'_2 + \sum_{k=1}^n a_{2k} b'_k \\ \vdots & \vdots & & \vdots & \vdots \\ \sum_{k=1}^n a_{nk} b_{k1} & \sum_{k=1}^n a_{nk} b_{k2} & \dots & \sum_{k=1}^n a_{nk} b_{kn} & a'_n + \sum_{k=1}^n a_{nk} b'_k \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} M_1 M_2 & a_1 + M_1 b_1 \\ 0 & 1 \end{pmatrix}$$

Similarly $BA = \begin{pmatrix} M_2 M_1 & b_1 + M_2 a_1 \\ 0 & 1 \end{pmatrix}$ and $AB \in X$
 $BA \in X$.

Let 11 denote the $n \times n$ identity matrix. $\text{aff}(11, 0) \in X$. Notice that $\text{aff}(11, 0)$ acts as an identity element:

$$\begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} M & a \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} M & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} M & a \\ 0 & 1 \end{pmatrix}$$

And the inverse of $\text{aff}(M, a)$ is $\text{aff}(M^{-1}, -M^{-1}a)$ hence

$$\text{aff}(M, a) \cdot \text{aff}(M^{-1}, -M^{-1}a) = \begin{pmatrix} MM^{-1} & a + M(-M^{-1}a) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

(R1) Since the set X is stable under multiplication, there exists an identity and for each element, there exists an inverse, X is a group under multiplication.

A The Euclidean Group as a Lie Group

Consider the set of pairs $X = \{(b, B) \mid b \in T(n), B \in O(n)\}$

And the composition of an element $x \in X$ with $a \in \mathbb{R}^n$ to be

$$(A1) \quad x \circ a = (b, B) \circ a = Ba + b$$

Additionally the composition of $x, y \in X$ with $x = (b, B), y = (b', B')$

$$(A2) \quad x \cdot y = (b, B) \cdot (b', B') = (b + Bb', BB')$$

This structure defines the Euclidean group.

Let us now construct a set

$$K := \{\text{aff}(B, b) \mid \forall B \in O(n), \forall b \in T(n)\}$$

Each element in K is a $(n+1) \times (n+1)$ matrix. the multiplication of

any 2 matrices in K gives us a result analogous to (A2). Namely

$$\text{aff}(B, b) \text{aff}(B', b') = \begin{pmatrix} B & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} BB' & b + Bb' \\ 0 & 1 \end{pmatrix} \in K$$

and the application of any element in K with a vector $\begin{pmatrix} x \\ 1 \end{pmatrix}$ of \mathbb{R}^{n+1} (appended with a 1 on its $(n+1)$ th entry) gives us a result analogous to (A1). Specifically

$$\text{aff}(B, b) \circ \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} B & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Bx + b \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$$
 where $x \in \mathbb{R}^n$

As $B \in O(n)$, B^{-1} exists also in $O(n)$. Therefore, we have from (R1)

that K forms a group. Additionally since any $k \in K$, $k \in GL(n+1)$

K is a subgroup of $GL(n+1)$

By our construction, for each pair $(b, B) \in E(n)$ there exists $\text{aff}(B, b) \in K$ bijectively, making K a faithful representation of the Euclidean group.

Thereby, we have that the Euclidean group can be faithfully represented as a Linear Lie group.

(B) The Poincaré group as a Lie group

Let us construct a set $P = \{(\Lambda, b) \mid \Lambda \in \mathcal{L}, b \in T(4)\}$

where \mathcal{L} denotes the Lorentz group.

The composition of any $p \in P$ with $x \in \mathbb{R}^4$ is given by:

$$(B1) \quad p \circ x = (\Lambda, b) \circ x = \Lambda x + b$$

And the composition of any two elements in P is given by:

$$(B2) \quad p \circ p' = (\Lambda, b)(\Lambda', b') = (b' + \Lambda b', \Lambda \Lambda')$$

This structure defines the Poincaré group.

Now let us construct the set $P_{\text{aff}} := \{\text{aff}(\Lambda, b) \mid \forall \Lambda \in \mathcal{L}, b \in T(4)\}$

Each element in P_{aff} is a 5×5 matrix. The multiplication of any 2 matrices in P_{aff} gives us a result analogous to (B2) namely:

$$\text{aff}(\Lambda, b) \text{aff}(\Lambda', b') = \begin{pmatrix} \Lambda & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Lambda \Lambda' & b + \Lambda b' \\ 0 & 1 \end{pmatrix}$$

and the application of $\text{aff}(\Lambda, b)$ on a vector $\begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{R}^5$, ($x \in \mathbb{R}^4$), gives us a result analogous to (B1) namely:

$$\text{aff}(\Lambda, b) \circ \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \Lambda x + b \\ 1 \end{pmatrix}$$

Since $\Lambda \in \mathcal{L}$, Λ^{-1} exists also in \mathcal{L} . Therefore we have from (R1) that P_{aff} forms a group. Additionally since $\forall p \in P_{\text{aff}}, p \in \text{GL}(5)$, P_{aff} is a subgroup of $\text{GL}(5)$.

By our construction of P_{aff} , for each pair $(\Lambda, b) \in P$ there exists $\text{aff}(\Lambda, b) \in P_{\text{aff}}$ bijectively, making P_{aff} a faithful representation of the Poincaré group.

Hence we have that the Poincaré group can be faithfully represented as a Linear Lie group.