

Exercise 1.2.12

Let a, b be elements of $Z(G)$

Clearly b is an element of G , so

$ab = ba$ by def of center.

$\therefore Z(G)$ is Abelian group.

By def of identity, $\forall g \in G, eg = ge$

$\Rightarrow e \in Z(G)$ i.e. $Z(G)$ has an identity

$Z(G)$ is Abelian group

$\Rightarrow \forall g \in G, (ab)g = a(bg) = a(gb) = (ag)b = g(ab)$ i.e. $(ab)g = g(ab)$

$\Rightarrow Z(G)$ is a group with map $Z(G) \times Z(G) \rightarrow Z(G)$

Also,

$\forall g \in G, ag = ga \Leftrightarrow a^{-1}aga^{-1} = a^{-1}gaa^{-1} \Leftrightarrow ga^{-1} = a^{-1}g.$

$\Rightarrow a^{-1} \in Z(G)$

In conclusion, $Z(G)$ is subgroup of G .

Name: Haruki Tsunekawa Student ID: 062200990

By def of center,

$$z(G)[g] = [g]z(G) \quad \text{for any } g \in G.$$

Using Prop. 1.2.9.

$Z(G)$ is a normal subgroup of G . \square

Exercise 1.2.15

1) ϕ is a group homomorphism

$$\Rightarrow \phi(e_G) = \phi(e_G)\phi(e_G) \quad \dots \textcircled{1} \quad (\because e_G = e_G e_G)$$

By def of the identity,

$$e_{G'} \phi(e_G) = \phi(e_G) \quad \dots \textcircled{2}$$

$$\textcircled{1}, \textcircled{2} \Rightarrow \phi(e_G)\phi(e_G) = e_{G'}\phi(e_G)$$

$$\Rightarrow e_{G'} = \phi(e_G) \quad (\because \text{Ex 1.1.3, 3})$$

$$e_{G'} = \phi(e_G) = \phi(aa^{-1}) = \phi(a)\phi(a^{-1}) \quad \dots \textcircled{3}$$

By def of the inverse,

$$e_{G'} = \phi(a)(\phi(a))^{-1} \quad \dots \textcircled{4}$$

$$\textcircled{3}, \textcircled{4} \Rightarrow (\phi(a))^{-1} = \phi(a^{-1})$$

2) $\phi(G_0)$ has the identity and the inverses ($\because 1$)

Let a, b, c be elements of G_0 .

$$\begin{aligned}\phi(a)(\phi(b)\phi(c)) &= \phi(a)\phi(bc) = \phi(a(bc)) \\ &= \phi((ab)c) \quad (\because a, b, c \in G_0) \\ &= \phi(ab)\phi(c) \\ &= (\phi(a)\phi(b))\phi(c)\end{aligned}$$

So, $\phi(G_0)$ satisfies associativity.

In conclusion, $\phi(G_0)$ is a subgroup of G'

3) I'll prove "Ker(ϕ) is a normal subgroup of G " by following two steps.

Step 1: Ker(ϕ) is a subgroup of G .

Step 2: Ker(ϕ) is a normal subgroup of G .

Step 1

Let a, b, c be elements of Ker(ϕ)

$$\phi(e_G) = e_{G'} \Rightarrow e_G \in \text{Ker}(\phi)$$

$$\phi(a^{-1}) = (\phi(a))^{-1} \Rightarrow \phi(a^{-1}) = e_{G'}^{-1} = e_{G'} \Rightarrow a^{-1} \in \text{Ker}(\phi)$$

$$\phi(ab) = \phi(a)\phi(b) = e_{G'}e_{G'} = e_{G'} \Rightarrow ab \in \text{Ker}(\phi)$$

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Clearly $a(bc) = (ab)c$ ($\because a, b, c \in G$)

Thus $\phi(G_0)$ is a subgroup of G .

Step 2

Let g be an element of G .

$$\begin{aligned}\phi(gag^{-1}) &= \phi(ga)\phi(g^{-1}) = \phi(g)\phi(a)\phi(g^{-1}) \\ &= \phi(g)\phi(a)(\phi(g))^{-1} \\ &= \phi(g)e_{G'}(\phi(g))^{-1} = e_{G'} \quad (\text{for any } a \in \text{Ker}(\phi))\end{aligned}$$

$\Rightarrow gag^{-1} \in \text{Ker}(\phi)$ for any $a \in \text{Ker}(\phi)$

$\Rightarrow \text{Ker}(\phi)$ is a normal subgroup of G . □

memo

4)

$$G / \text{Ker}(\phi) = \{ \underbrace{[a]_{\text{Ker}(\phi)}}_1 \mid a \in G \}.$$

$$\{ b \mid a \sim b \} \Leftrightarrow \exists c \in \text{Ker}(\phi) \text{ s.t. } ba^{-1} = c. \quad a, b \in G.$$

Show $G / \text{Ker}(\phi) \simeq \phi(G)$

\Leftrightarrow There exists a bijective map $\tilde{\phi}$ s.t. $\tilde{\phi} : G / \text{Ker}(\phi) \rightarrow \phi(G)$

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It is enough to show that $\tilde{\phi}$ is homomorphism and bijective.

$$\begin{aligned}\tilde{\phi}([a]_{\ker(\phi)} [b]_{\ker(\phi)}) &= \tilde{\phi}([ab]_{\ker(\phi)}) \quad (\text{By def of } G/\ker(\phi)) \\ &= \phi(ab) \quad \dots (i) \quad (\text{By def of } \tilde{\phi})\end{aligned}$$

$$\begin{aligned}\tilde{\phi}([a]_{\ker(\phi)}) \tilde{\phi}([b]_{\ker(\phi)}) &= \phi(a) \phi(b) = \phi(ab) \quad \dots (ii) \\ & \quad (\because \phi \text{ is homomorphism})\end{aligned}$$

(i), (ii) $\Rightarrow \tilde{\phi}$ is homomorphism.

$$\text{If } \ker \phi(a) = \ker \phi(b) \Rightarrow [a]_{\ker \phi} = [b]_{\ker \phi} \Rightarrow \tilde{\phi}([a]_{\ker \phi}) = \tilde{\phi}([b]_{\ker \phi})$$

$$\begin{aligned}\text{Also if } [a]_{\ker \phi} = [b]_{\ker \phi} &\Rightarrow \phi([a]_{\ker \phi}) = \phi([b]_{\ker \phi}) \Rightarrow \phi(k_1) \phi(a) = \phi(k_2) \phi(b) \\ &\Rightarrow \phi(a) = \phi(b) \quad (\text{for any } k_1 \in [a]_{\ker \phi}, k_2 \in [b]_{\ker \phi})\end{aligned}$$

So, $\tilde{\phi}$ is injective.

For all $\phi(a)$, there exists some coset including a because we can divide all elements in G into some cosets. This means $\tilde{\phi}$ is surjective.

Therefore $\tilde{\phi}$ is bijective

□