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Let L be a Lie algebra over \mathbb{K} , and let us consider the map ad defined by

$$\text{ad}: L \ni X \mapsto \text{ad}_X \in \mathcal{L}(L)$$

with $\text{ad}_X(Y) = [X, Y]$. The following properties hold for any $X, X' \in L$ and $\alpha \in \mathbb{K}$, and next is the proof that ad holds the 3 properties.

1) $\text{ad}_0 = 0$

2) $\text{ad}_{x+\alpha x'} = \text{ad}_x + \alpha \text{ad}_{x'}$

3) $\text{ad}_{[X,Y]} = \text{ad}_X \text{ad}_Y - \text{ad}_Y \text{ad}_X = [\text{ad}_X, \text{ad}_Y]$

1) When Y is arbitrary,

$$\text{ad}_0(Y) = [0, Y]$$

$$= 0 \cdot Y - Y \cdot 0$$

$$= 0$$

2) When Y is arbitrary,

$$\begin{aligned}
\text{ad}_{x+\alpha x'}(Y) &= [x + \alpha x', Y] \\
&= (x + \alpha x') Y - Y(x + \alpha x') \\
&= XY - YX + \alpha(XY - YX) \\
&= \text{ad}_x(Y) + \alpha \text{ad}_{x'}(Y)
\end{aligned}$$

3) From the Jacobi identity,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

so,

$$\begin{aligned}
\text{ad}_{[x,y]}(z) &= -[z, [x, y]] \\
&= [x, [y, z]] + [y, [z, x]] \\
&= (\text{ad}_x \text{ad}_y - \text{ad}_y \text{ad}_x)(z) \\
&= [\text{ad}_x, \text{ad}_y](z)
\end{aligned}$$

Since z is arbitrary, one gets 3)

$$\begin{aligned}
A(\text{so}, & \quad [\text{ad}_x(Y), z] + [Y, \text{ad}_x(z)]) \\
&= [[x, Y], z] + [Y, [x, z]] \\
&= -[z, [x, Y]] - [Y, [z, x]]
\end{aligned}$$

From the Jacobi identity,

$$-[z, [x, y]] - [y, [z, x]] = [x, [y, z]]$$

$$\text{so, } [\text{ad}_x(y), z] + [y, \text{ad}_x(z)] \\ = [x, [y, z]]$$

$$= \text{ad}_x([y, z])$$

$$\Rightarrow \text{ad}_x([y, z]) = [\text{ad}_x(y), z] + [y, \text{ad}_x(z)]$$