Stabilizers and orbits

For any transformation group G of a set X one has:

1) The stabilizer G_x of x is a subgroup of G, for any $x \in X$
2) If $x' \in O_x$(the orbit of x) then $G_{x'} \simeq G_x$

Proof of the first statement: To prove G_x is a subgroup of G, we must check the properties of the group.

Let $y, z \in G_x$. Then,

$$(yz) \cdot x = y \cdot (z \cdot x) = y \cdot x = x$$

meaning that $yz \in G_x$. Thus, G_x is closed under associativity.

We know the identity element $e \in$ G, and $e \cdot x = x$. Then it follows that $e \in G_x$.

Thus, proving the existence of an identity element in G_x.

Let $z \in G_x$. Then,

$$x = e \cdot x = (z^{-1}z) \cdot x = z^{-1} \cdot (z \cdot x) = z^{-1} \cdot x.$$

It follows that $z^{-1} \in G_x$, proving the existence of the inverse in G_x.

Thus, G_x is a subgroup of G for any $x \in X$. ■

Proof of the second statement: For $x' \in O_x$ we know that there is $y \in G$ such that $x' = y \cdot x$.

We define a bijective map $\phi : G_x \to G_{x'}$ by $\phi(a) = yay^{-1}$. It can be easily observed that if a stabilizes x, then yay^{-1} stabilizes x'.

To show that the map is injective, we assume $\phi(a_1) = \phi\ (a_2)$ for elements $a_1, a_2 \in G_{x'}$ and show that $a_1 = a_2$:

$$\phi(a_1) = ya_1y^{-1}, \phi\ (a_2) = ya_2y^{-1}$$

$$ya_1y^{-1} = ya_2y^{-1}$$

$$a_1 = a_2$$

Thus, the map ϕ is injective.

Next, we take $z' \in G_{x'}$ and set $z = y^{-1}z'y$. Then, one checks that $z \in G_x$ and $\phi(z) = z'$, showing that ϕ is surjective. As a consequence, ϕ is bijective, and clearly a homomorphism. It is thus an isomorphism. ■