Open and closed sets in topological spaces

Groups and their representations - Report

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January 17, 2023

1 Introduction

In a topological space (\mathcal{M}, τ) all elements of τ are defined to be open sets, and their complement $\overline{V} = \mathcal{M} \setminus V$ are defined to be closed sets $(\forall V \in \tau)$. A natural question can be whether there exists a set that is both open and closed. An example to these kind of sets can be seen in Figure 1. The aim of this report is to investigate these kind of sets.

2 Observations

Firstly, for the sake of brevity, let us call sets that are both open and closed *clopen* sets. In other words, a set $V \subset \mathcal{M}$ is *clopen*, if $V, \overline{V} \in \tau$ ($V \in \tau \Rightarrow V$ is open, $\overline{V} \in \tau \Rightarrow V$ is closed). Observe that $\{p_1\}$ and $\{p_2, p_3\}$ are clopen sets in Figure 1. Only by considering the definitions and some simple cases, one can make the following observations.

Observation 1 For a topological space (\mathcal{M}, τ)

- 1. \emptyset and \mathcal{M} are always clopen. This follows from the fact that by definition \emptyset and \mathcal{M} are open sets and that they are the complement of each other (so they are closed sets too, which is the definition of clopen set).
- 2. All subset of \mathcal{M} is clopen if and only if τ contains all subsets of \mathcal{M} .

Proof: if all subset of \mathcal{M} is contained by τ , then all subset of \mathcal{M} is open (by definition). Also, since the complement of any subset of \mathcal{M} is another subset of \mathcal{M} , the complement of any subset of \mathcal{M} is open, thus any subset of \mathcal{M} is closed. Therefore, if all subset of \mathcal{M} is contained by τ , then all subset of \mathcal{M}



Figure 1: **Open and closed sets.** In this figure $\{p_1\}$ is an open set (since it is in τ), and its complement $\{p_2, p_3\}$ is a closed set. Observe that since $\{p_2, p_3\}$ is also contained by τ , $\{p_2, p_3\}$ is also open and $\{p_1\}$ is thus also closed, so $\{p_1\}$ and $\{p_2, p_3\}$ are both open and closed.

is clopen.

Now suppose that all subset of \mathcal{M} is clopen, but not all subset of \mathcal{M} is contained by τ . This means that there exists $V \subset \mathcal{M}$ that is not in τ . But that would mean that V is not an open set, thus V cannot be clopen. Hence, if \mathcal{M} only contains clopen sets, then all subset of \mathcal{M} is contained by τ (proof by contradiction).

3. If V is clopen, then $C = \{V, \overline{V}\}$ is an open cover of \mathcal{M} . This follows from the fact that any element of \mathcal{M} can be found either in V or in its complement (by definition), and that both V and \overline{V} are open sets (this is exactly what "V is clopen" means).

3 Hausdorff property

Next let us investigate the relationship between clopen sets and the Hausdorff property. Recall that a topological space (\mathcal{M}, τ) is Hausdorff if for any $p_1, p_2 \in \mathcal{M}$ with

	contains clopen sets	does not contain clopen sets
	(other than \emptyset and \mathcal{M})	(other than \emptyset and \mathcal{M})
Hausdorff	$\mathcal{M} = \{p_1, p_2\}$?
	$\tau = \{\mathcal{M}, \emptyset, \{p_1\}, \{p_2\}\}$	
not Hausdorff	$\mathcal{M} = \{p_1, p_2, p_3\}$	$\mathcal{M} = \{p_1, p_2, p_3\}$
	$\tau = \{\mathcal{M}, \emptyset, \{p_1\}, \{p_2, p_3\}\}$	$\tau = \{\mathcal{M}, \emptyset, \{p_1\}, \{p_2\}, \{p_1, p_2\}\}$

 $p_1 \neq p_2$ there exists $V_1, V_2 \in \tau$ such that $p_1 \in V_1, p_2 \in V_2$, and $V_1 \cap V_2 = \emptyset$.

Table 1: Clopen sets and Hausdorff property.



Figure 2: A Hausdorff topological space containing clopen sets (other than \emptyset and \mathcal{M}). Observe that it is possible to separate p_1 and p_2 using open sets (so it is Hausdorff). Also, observe that $\{p_1\}$ and $\{p_2\}$ are clopen.

Table 1 provides one-one example for a Hausdorff topological space containing clopen sets (other than \emptyset and \mathcal{M}), a non-Hausdorff topological space containing clopen sets (other than \emptyset and \mathcal{M}), and a non-Hausdorff topological space not containing clopen sets (other than \emptyset and \mathcal{M})¹. An explanation for each case can be seen in Figure 2, 3, and 4. However, one can notice that for the entry of a Hausdorff topological

¹Note: This table provides two solutions for Exercise 3.1.4. (in the bottom row).



Figure 3: A not Hausdorff topological space containing clopen sets (other than \emptyset and \mathcal{M}). Observe that p_2 and p_3 cannot be separated using open sets (so it is not Hausdorff). Also, observe that $\{p_1\}$ and $\{p_2, p_3\}$ are clopen.

space not containing clopen sets (other than \emptyset and \mathcal{M}) it is not straightforward to provide an (easy) example. In fact, it turns out that if $|\mathcal{M}| < \infty$, then no such set exists, which will follow from a following, stronger proposition. But, to prove that proposition, we need to prove the following lemma first.

Lemma 1 Let (\mathcal{M}, τ) be a Hausdorff topological space. For any element $p \in \mathcal{M}$, if the subset containing only this element is not an open set, then any finite subset of \mathcal{M} containing p is not an open set either (i.e. for $\forall p \in \mathcal{M}$, if $\{p\} \notin \tau$ and $p \in V \subset \mathcal{M}$ with $|V| < \infty$, then $V \notin \tau$).

<u>Proof:</u> I am going to use proof by mathematical induction. Let us assume that (\mathcal{M}, τ) is Hausdorff, and that for a $p \in \mathcal{M}$, $\{p\} \notin \tau$.

Next, let us check the subsets of \mathcal{M} containing p. Let us start with the ones containing the least number of elements (the most basic cases)

When the subset has one element: there is only one subset of M that has one element and contains p, namely {p}, which is not an open set by our basic assumption ⇒ no subset of M containing one element is open if it contains p.



Figure 4: A not Hausdorff topological space not containing clopen sets (other than \emptyset and \mathcal{M}). Observe that p_1 and p_3 cannot be separated using open sets (so it is not Hausdorff). Also, observe that there are no clopen sets in this topological space (except for \emptyset and \mathcal{M}): neither of the other open sets ($\{p_1\}, \{p_2\}, \{p_1, p_2\}$) are closed, since their complements ($\{p_2, p_3\}, \{p_1, p_3\}, \{p_3\}$) are not open sets, thus they are not clopen.

When the subset has two elements: this is a set containing p and another random element of M, that is a set {p,p'} with p' ∈ M and p ≠ p'. Now let us assume that this can be an open set, that is ∃p' ≠ p in M for which {p,p'} ∈ τ. However, the topological space is Hausdorff ⇒ ∃V, V' ∈ τ with p ∈ V, p' ∈ V', and V ∩ V' = Ø. Then p' ∉ V ⇒ {p,p'} ∩ V = {p} ⇒ {p} ∈ τ, which is a contradiction, thus {p,p'} ∉ τ for any p' ∈ M. ⇒ no subset of M containing two elements is open if it contains p.

We have seen that the statement we want to prove works for subsets containing one or two elements. Next let us assume that it also holds for subsets containing no more than k elements, and see if it holds for any set containing k + 1 elements. For this let us assume that the set $\{p, p_1, p_2, ..., p_k\}$ is open and contains distinct elements (in other words, this is a set containing k + 1 distinct elements that also contains p), but any proper subset of it containing p is not open. Since the topological space is Hausdorff, for $1 \leq i \leq k$, $\exists V, V_i \in \tau$ with $p \in V$, $p_i \in V_i$ ($p_i \neq p$), and $V \cap V_i = \emptyset \Rightarrow p_i \notin V \Rightarrow \{p, p_1, p_2, ..., p_k\} \cap V \subset \{p, p_1, p_2, ..., p_{i-1}, p_{i+1}, ..., p_k\} \Rightarrow a$ subset of $\{p, p_1, p_2, ..., p_{i-1}, p_{i+1}, ..., p_k\}$ is an open set, which is a contradiction, since any subset of $\{p, p_1, p_2, ..., p_{i-1}, p_{i+1}, ..., p_k\}$ is a proper subset of $\{p, p_1, p_2, ..., p_k\}$ containing p (and no proper subset of $\{p, p_1, p_2, ..., p_k\}$ containing p is supposed to be open).

From these, by mathematical induction, one gets that if $\{p\} \notin \tau$, then any finite subset of \mathcal{M} containing p is not open. However, since p is an arbitrary element of \mathcal{M} , this proof can be applied to any element of \mathcal{M} , thus we have proven the statement we wanted to prove (i.e. for $\forall p \in \mathcal{M}$, if $\{p\} \notin \tau$ and $p \in V \subset \mathcal{M}$ with $|V| < \infty$, then $V \notin \tau$).

Remark 1 The reason why we need the V subset of \mathcal{M} to be finite in Lemma 1 is that the recursion (by which we prove that V is not an open set) can only be performed a finite number of times, which means that V can have an arbitrarily big number of elements, but not an infinite number of elements so that the proof still works.

Now we can move on to the proposition.

Proposition 1 If a topological space (\mathcal{M}, τ) is Hausdorff and $|\mathcal{M}| < \infty$, then τ contains any subset of \mathcal{M} .

<u>Proof:</u> I will use proof by contradiction. Let $\mathcal{M} = \{p_1, ..., p_n\}$ be Hausdorff, and let us assume that there exists $p_j \in \mathcal{M}$ such that $\{p_j\} \notin \tau$. By Lemma 1 we have that if $\{p_j\} \notin \tau$, then any subset of \mathcal{M} containing $\{p_j\}$ is not in τ too for any $p_j \in \mathcal{M}$ (since \mathcal{M} is finite, any subset of it is finite too). However, since \mathcal{M} is a subset of itself, if such a p_j exists, then \mathcal{M} cannot be in τ , which is a contradiction ($\mathcal{M} \in \tau$ by definition of the topological space). Thus, $\forall p_j \in \mathcal{M}, \{p_j\} \in \tau$. But, from this we get that since $\forall V = \{p_{i_1}, p_{i_2}, ..., p_{i_k}\} \subset \mathcal{M}$ can be written as $V = \bigcup_{j=1}^k \{p_{i_j}\}$, any subset of \mathcal{M} is open, i.e. $\forall V \in \tau$ because elements of τ are stable under union.

By Proposition 1 and the second statement of Observation 1, one can see that if (\mathcal{M}, τ) is Hausdorff and $|\mathcal{M}| < \infty$, then it contains only clopen sets. This also means that we cannot find any topological space with finite number of elements that can fit into the cell of Table 1 that is marked with "?", like we did in the other three cases.

However, the following (infinite) topological space is Hausdorff but does not contain clopen sets (other than \emptyset and \mathcal{M}). Consider $\mathcal{M} = \mathbb{R}$ and the basis $\mathcal{B} = \{I = (x - r, x + r) | x, r \in \mathbb{R} \text{ with } r > 0\}$. This is Hausdorff, since for any x_1, x_2 , we can choose two intervals $I_1 = \left(x_1 - \frac{|x_1 - x_2|}{4}, x_1 + \frac{|x_1 - x_2|}{4}\right)$ and $I_2 = \left(x_2 - \frac{|x_1 - x_2|}{4}, x_2 + \frac{|x_1 - x_2|}{4}\right)$, which contain x_1 and x_2 , respectively, and which are open, but disjoint. However, the topological space does not contain clopen sets (apart from \mathbb{R} and \emptyset) for the following reason: if we have $V = (x_0, x_1) \cup (x_2, x_3) \cup (x_4, x_5) \cup ... \in \tau$ with $x_0 \leq x_1 \leq x_2 \leq ...$, then $\mathbb{R} \setminus V = (-\infty, x_0] \cup [x_1, x_2] \cup [x_3, x_4] \cup ...$, thus the complement of V is the union of at least one closed or half-open interval. However, neither a closed, nor a halfopened interval can be produced as a union or finite intersection of open intervals, thus $\overline{V} \notin \tau$ for any $V \in \tau$ ($V \neq \emptyset$, $V \neq \mathbb{R}$). Therefore, this topological space is Hausdorff but does not contain clopen sets (other than \emptyset and \mathcal{M}). This also means that this topological space can be put into the cell of Table 1 marked with "?".

Remark 2 Observe that in the topological space in the previous example, sets only containing one distinct point are not open sets (e.g. $\{3\} \notin \tau$, since $\{3\} = [3,3]$, which is a closed interval, and we have seen that closed intervals cannot be open sets in that topological space), but infinite subsets of \mathcal{M} containing these points can be open (e.g. $\{3\} \subset (2,4) \subset \mathbb{R}$, where (2,4) and \mathbb{R} are both open). This provides an example supporting the claim of Remark 1.

4 Conclusion, further questions

In this report some basic properties of clopen sets (i.e. sets that are both open and closed) were analyzed, as well as their relationship with Hausdorff property. It turned out that if a Hausdorff topological space (\mathcal{M}, τ) has a finite number of elements, then not only it has at least one clopen set (other than the trivial cases), but all of the sets in it are clopen. However, if this Hausdorff topological space has an infinite number of elements, then it can have zero clopen sets (other than the empty set and itself), for which an example was provided in the report.

Finally, I would like to provide a few additional/follow-up questions, that are related to this topic, but were not discussed in this report (but I think are questions that could also be interesting)

- 1. What about sets that are neither open nor closed?
- 2. Can the topological space (\mathbb{N}, τ) be Hausdorff while containing only the two trivial clopen sets? (This question can be interesting, since for a finite set the answer is 'no', but for an uncountably infinite set, such as \mathbb{R} , the answer is 'yes', so we could ask what is the answer in the case of a countably infinite set, which is kind of between the two previous cases.)

5 References

1. Cumulative notes for the course