Reminder XII

- Let L be endowed with the standard basis. A root α is *positive* if the first non-zero entry of $(\alpha(H_{d_0}), \dots, \alpha(H_1))$ is positive. Then $R = R_+ \cup R_-$, $R_+ \cap R_- = \emptyset$, and $\beta > \alpha$ if $\beta - \alpha \in R_+$ (lexicographic order).

  positive root / negative roots

- A positive root is *simple* if it is not a positive linear combination of other positive roots.

- There exist *d_0 linearly independent simple roots*, all other roots are obtained by linear combinations with coefficients in $\mathbb{Z}_+$ or in $\mathbb{Z}_-$.

- Let $(\mathcal{V}, h)$ be a finite dimensional representation of a complex semi-simple Lie algebra L. If $\exists\, v \in \mathcal{V}$, $v \neq 0$ with $h(H)\, v = \mu(H)\, v \quad \forall\, H \in L_0$ and $\mu(H) \in \mathbb{C}$, then $\mu \in L_0^*$ is called a *weight*, and v a *weight vector*. The dimension of $L_\mu := \{ v \in \mathcal{V} \mid h(H)\, v = \mu(H)\, v, \ \forall H \in L_0 \}$ is the *multiplicity* of μ.

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• Weights and roots are quite similar, but roots are more intrinsic since (L, ad) is a faithful representation.

• If L has the standard basis, $\mu_j := \mu(H_j) \in \mathbb{R}$, $\overbrace{\mu + k\alpha}^{\in \mathbb{R}^{do}}$ is a weight whenever $\underbrace{\overbrace{h(E_\alpha)}^{=:\mathcal{E}_\alpha}}_{\text{raising or lowering operator.}}{}^{k}\, v \neq 0$, $k \in \mathbb{Z}$. which k ?

• __Thm__ : 1) $-2\frac{\nu \cdot \alpha}{\|\alpha\|^2} =: N \in \mathbb{Z}$, 2) $\mu + k\alpha$ is a weight $\forall k \in [0, N]$, (mirror symmetry)

3) $\text{Span}(v, \mathcal{E}_\alpha v, \mathcal{E}_\alpha \mathcal{E}_\beta v, \ldots) = V$ $\forall v$ root vector
$\in R$

4) The number of weights with multiplicity is equal to $\dim(V)$,

5) weights provide the eigenvalues of $h(H)$, $\forall H \in L_0$.

6) Knowing (μ_{max}, v_{max}) provide everything through the lowering operators.
of multiplicity 1

The set of μ_{max} can be indexed, and the corresponding irreducible representation can be computed.