Reminder XI

Lie algebra — symmetric and bilinear

- Killing form $K : L \times L \to \mathbb{C}$, $K(X,Y) := Tr(ad_X\, ad_Y)$, $X, Y \in L$.

structure coeficients

If $\{X_1, \dots, X_d\}$ is a basis of L, $g_{jk} := K(X_j, X_k) = \sum_{r,s=1}^{d} C_{jr}^{s} C_{ks}^{r}$

matrix

If $Det((g_{jk}))$, K is non-degenerate.

- 3 results: 1) L a semi-simple Lie algebra $\Leftrightarrow$ K non-degenerate,

no loss of information

2) L semi-simple Lie algebra $\Rightarrow$ adjoint representation faithful

$K(X,X) < 0 \;\; \forall X \neq 0$

3) G connected semi-simple group is compact $\Leftrightarrow$ K negative definite

adjoint representation

- ⚠ Not all $A \in M_n(\mathbb{C})$ are diagonalizable. ⇝ Jordan form ⇝ generalized eigenvectors

- Cartan subalgebra L_0 of a complex semi-simple Lie algebra:

↳ always in the sequel

maximal Abelian subalgebra with ad_X diagonalizable, $\forall X \in L_0$.

Rank = dimension of L_0, denoted d_0 if $\dim(L) = d$.

- Root α : non-zero linear map on L_0 s.t. $\exists\, Y_\alpha \in L$ with

denoted by L_0^*

$ad_X(Y_\alpha) = \alpha(X)\, Y_\alpha \quad \forall X \in L_0$. $\mathcal{R}$ the set of roots.

- If $\{X_1, \dots, X_{d_0}\}$ is a basis of L_0, α uniquely defined

by $\alpha(X_1), \dots, \alpha(X_{d_0})$.

- If $\alpha \in \mathcal{R}$, $-\alpha \in \mathcal{R}$; $\exists\ d-d_0$ different roots $\} \Rightarrow d-d_0$ is even.

- $\exists$ standard basis of L : $\{H_1, \dots, H_{d_0}, E_\alpha, E_{-\alpha}, \dots \quad, E_\mu, E_{-\mu}\}$

with the properties : 1) $[H_j, H_k] = 0$, 2) $[H_j, E_\alpha] = \underbrace{\alpha(H_j)}^{\in \mathbb{R}} E_\alpha$,

3) $[E_\alpha, E_\beta] = \dots$ and $(g_{jk}) = \left(\begin{array}{c|c} \begin{smallmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix} 0 & 1 & & & \\ 1 & 0 & & & 0 \\ & & 0 & 1 & \\ 0 & & 1 & 0 & \\ & & & & \ddots \end{smallmatrix} \end{array}\right)$.

- Examples of $su(2)_{\mathbb{C}}$ and $su(3)_{\mathbb{C}}$.