Nagoya University, G30 program Fall 2022

Calculus I Instructor : Serge Richard

Final examination Life is beautiful, math is part of it. Good luck !

Exercise 1 *Compute the derivatives of the following functions:*

$$(i)\ x \mapsto \frac{x-1}{x^2+1} \qquad (ii)\ x \mapsto x\sin(1/x) \qquad (iii)\ x \mapsto \sqrt{x+\sqrt{x}}.$$

Exercise 2 *Compute the following integrals:*

$$(i)\ \int_3^4 \frac{1}{(x-2)(x-1)}\,\mathrm{d}x \qquad (ii)\ \int x\,2^x\,\mathrm{d}x \qquad (iii)\ \int_0^{\sinh(1)} \sqrt{1+x^2}\,\mathrm{d}x.$$

Exercise 3 *Consider $(a_j)_{j=1}^{\infty}$ a sequence of real numbers.*

(i) What does it mean that sequence is convergent ? (give the definition)

(ii) What does it mean that the corresponding series converges ? (give the definition)

(iii) What does it mean that the corresponding series converges absolutely? (give the definition)

(iv) If $a_j = \frac{(-1)^j}{5^j}$, is the sequence convergent, is the corresponding series convergent, is the corresponding series absolutely convergent ? Justify briefly your answers.

Exercise 4 *Consider the function $f : (0,\infty) \setminus \{1\} \ni x \mapsto \frac{1}{x(\ln(x))^2} \in (0,\infty)$. The following 4 questions are essentially independent.*

(i) Determine the critical point(s) of f, and its local minima and local maxima,

(ii) Sketch the function f,

(iii) Find an indefinite integral for f and compute $\int_0^{1/2} f(x)\,\mathrm{d}x$,

(iv) A Laurent series for the function f at 1 is an expansion of the form $f(1+y) = \sum_{j\in\mathbb{Z}} c_j\, y^j$ with $c_j \in \mathbb{R}$ and $|y|$ small enough but $y \neq 0$. Determine the coefficient c_{-2} for the Laurent series of f at 1. The starting point is to set $x = 1 + y$ for $|y|$ small enough but $y \neq 0$.

Final exam — Total 31 pts

Exercise 1 — 6 pts

i) $\left(\frac{x-1}{x^2+1}\right)' = \frac{1(x^2+1) - 2x(x-1)}{(x^2+1)^2} = \underline{\underline{\frac{-x^2+2x+1}{(x^2+1)^2}}}$. 2

ii) $(x \sin(1/x))' = \sin(1/x) - x\cos(1/x)\frac{1}{x^2}$

$= \underline{\underline{\sin(1/x) - \frac{1}{x}\cos(1/x)}}$. 2

iii) $\left((x + x^{1/2})^{1/2}\right)' = \underline{\underline{\frac{1}{2}(x + x^{1/2})^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right)}}$. 2

Exercise 2 — 9 pts

i) $\int_3^4 \frac{1}{(x-2)(x-1)}\,dx = \int_3^4 \left(\frac{1}{x-2} - \frac{1}{x-1}\right) dx$

$= \ln(x-2)\Big|_3^4 - \ln(x-1)\Big|_3^4$

$= \ln(2) - \ln(1) - \ln(3) + \ln(2)$

$= \underline{\underline{\ln(4/3)}}$. 3

ii) $\int x\, 2^x\, dx = \int x\, e^{x\ln(2)}\, dx =$

$= \frac{1}{\ln(2)} x\, e^{x\ln(2)} - \frac{1}{\ln(2)} \int e^{x\ln(2)}\, dx = \ldots$

$$= \frac{1}{\ln(2)} \, x \, e^{x \ln(2)} - \frac{1}{\ln(2)^2} \, e^{x \ln(2)} .$$ 3

iii) $\int_0^{\sinh(1)} \sqrt{1+x^2}\, dx$ set $x = \sinh(y)$, $dx = \cosh(y)\, dy$

$$= \int_0^1 \cosh(y)^2 \, dy$$

$$= \int_0^1 \frac{(e^y + e^{-y})^2}{4} \, dy$$

$$= \frac{1}{4} \int_0^1 (e^{2y} + e^{-2y} + 2) \, dy$$

$$= \frac{1}{4} \left(\frac{1}{2} e^{2y} - \frac{1}{2} e^{-2y} + 2y \right) \Big|_0^1$$

$$= \frac{1}{4} \left(\frac{1}{2} e^2 - \frac{1}{2} e^{-2} + 2 - 0 \right)$$

$$= \frac{1}{4} \left(2 + \sinh(2) \right) \quad \text{or} \quad \frac{1}{4} \left(2 + \frac{1}{2} e^2 - \frac{1}{2} e^{-2} \right) .$$

3

Exercise 3

7 pts

i) The sequence is convergent if $\exists\, a_\infty \in \mathbb{R}$ such that for any $\varepsilon > 0$, $\exists N$ with $|a_j - a_\infty| \leq \varepsilon$ $\forall j \geq N$. 2

ii) The series converges if the sequence of partial sum converges, namely if $\bar{S}_n := \sum_{j=1}^{n} a_j$ defines a convergent sequence. 1

iii) The series converges absolutely if $S_n := \sum_{j=1}^{n} |a_j|$ defines a convergent sequence. 1

iv) The sequence converges to 0, and thus is convergent, since $\forall \varepsilon > 0$, $\frac{1}{5j} < \varepsilon$ for j large enough. The series converges because it is an alternating series. The series is absolutely convergent since $\frac{|a_{j+1}|}{|a_j|} = \frac{5^j}{5^{j+1}} = \frac{1}{5} < 1 \quad \forall j$. By the ratio criterion, the series converges (which corresponds to the absolute convergence). 3

Exercise 4 — 9 pts

i) $f'(x) = \left(x \ln(x)^2\right)^{-1\,\prime} = \dfrac{-\left[\ln(x)^2 + 2x \ln(x) \frac{1}{x}\right]}{\left(x \ln(x)^2\right)^2}$

$= -\dfrac{\ln(x) + 2}{x^2 \ln(x)^3}$.

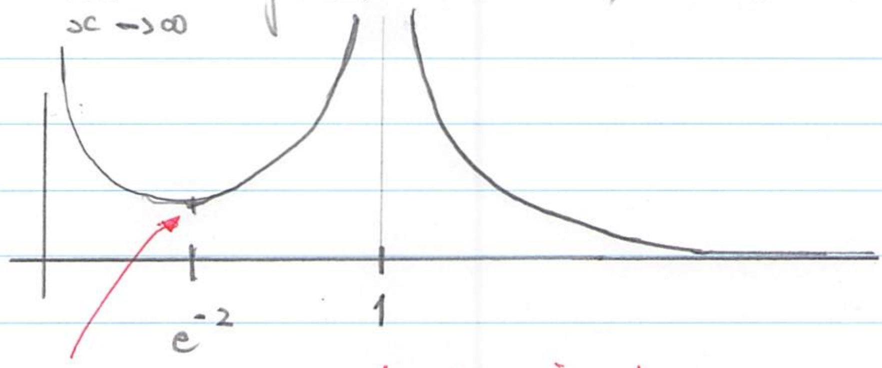
The critical point is given by $\ln(x) + 2 = 0$

$\Leftrightarrow \ln(x) = -2 \Leftrightarrow x = e^{-2} = \dfrac{1}{e^2}$. 2

Since $\lim_{x \searrow 0} f(x) = +\infty$ and $\lim_{x \nearrow 1} f(x) = +\infty$,

$x = e^{-2}$ is a local minimum of f. 1

ii) Since $\lim_{x \to \infty} f(x) = 0$, one has



not defined

2

$f(e^{-2}) = \left(e^{-2}(-2)^2\right)^{-1} = \dfrac{1}{4} e^2$.

iii) Observe that $\left((\ln(x))^{-1}\right)' = -(\ln(x))^{-2}\,\frac{1}{x}$

$= -(x \ln(x)^2)^{-1} = -f(x)$. Thus,

$$\int_0^{1/2} f(x)\,dx = -(\ln(x))^{-1}\Big|_0^{1/2} =$$

$$= -(\ln(1/2))^{-1} + \lim_{\varepsilon \searrow 0} (\ln(\varepsilon))^{-1}$$

$$= -(-\ln(2))^{-1} + 0 = \underline{\underline{1/\ln(2)}}.$$ 2

iv) $f(1+y) = \left((1+y)(\ln(1+y))^2\right)^{-1}$

Taylor expansion of $\ln(1+y)$ $= \left((1+y)\left(y - \frac{1}{2}y^2 + R_3(y,0)\right)^2\right)^{-1}$

$$= \left((1+y)\left(y^2 + \tilde{R}_3(y,0)\right)\right)^{-1}$$

$$= \left(y^2 + \tilde{\tilde{R}}_3(y,0)\right)^{-1}$$

$$= y^{-2}\left(1 + y^{-2}\tilde{\tilde{R}}_3(y,0)\right)^{-1}$$

$$= y^{-2}\left(1 + \hat{R}_1(y,0)\right)$$

$$= y^{-2} + \hat{\hat{R}}_{-1}(y,0)$$

$R_n(y,0)$ gives the behavior of the remainder.

$\Rightarrow C_{-2} = \underline{\underline{1}}.$ 2

It provides an information about the type of singularity.

名古屋大学大学院多元数理科学研究科