Homework 5

Exercise 1 *Find the equation of the tangent of the curve in* $\mathbb{R}^2$ *defined by the relation*

$$F(x,y) = x^2 - y^2 + 3xy + 12 = 0$$

at the point $(-4,2)$.

Exercise 2 *For* $n \in \mathbb{N}$ *let us set* $p_{\frac{1}{n}} : (0,\infty) \to \mathbb{R}$ *for the function defined by* $p_{\frac{1}{n}}(x) := x^{\frac{1}{n}}$. *If* $m \in \mathbb{N}$ *we also set* $p_{\frac{m}{n}} : (0,\infty) \to \mathbb{R}$ *by* $p_{\frac{m}{n}}(x) \equiv x^{\frac{m}{n}} := \big(x^m\big)^{\frac{1}{n}} = \big(x^{\frac{1}{n}}\big)^m$. *Finally, for* $q \in \mathbb{Q}_+$ *we define the function* $p_{-q} : (0,\infty) \to \mathbb{R}$ *by* $p_{-q}(x) \equiv x^{-q} := \frac{1}{x^q}$.

1) Show that the following equality holds:

$$p'_{\frac{1}{n}}(x) = \frac{1}{n} x^{\frac{1}{n}-1} \ .$$

For the proof you can use the equality

$$(a^n - b^n) = (a-b) \sum_{k=0}^{n-1} a^{n-k-1} b^k$$

for $a = (x+h)^{\frac{1}{n}}$ *and* $b = x^{\frac{1}{n}}$. *Other arguments which do not involve this formula are also possible.*

2) For $m,n \in \mathbb{N}$, *deduce that*

$$p'_{\frac{m}{n}}(x) = \tfrac{m}{n} x^{\frac{m}{n}-1} \ .$$

3) For any $q \in \mathbb{Q}_+$, *show that*

$$p'_{-q}(x) = -q x^{-q-1}.$$

Conclude that the equality $p'_q = q\, p_{q-1}$ *holds for any* $q \in \mathbb{Q}$.

Exercise 3 *Compute and simplify the derivative of the functions* $f : \mathbb{R} \to \mathbb{R}$ *defined for* $x \in \mathbb{R}$ *by*

$$a)\ \sin\big((2x^2-3)^2\big) \qquad b)\ \frac{(x+3)^3}{(2x-3)^2+1} \qquad c)\ \frac{1}{\sin^2(3x)+1}$$

Exercise 4 *Compute the following limits:*

(i) $\lim_{x\to 0} \frac{x-\sin(x)}{x^3}$,

(ii) $\lim_{x\to 0} \frac{x^2}{1+x-\mathrm{e}^x}$.