

SML  
 Differential Equations  
 and Dynamical Systems

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Problem 3.17 Teschl

$$i) \ddot{x} + 3\dot{x} + 2x = \sinh(t), \text{ homogeneous equation: } \ddot{x} + 3\dot{x} + 2x = 0$$

Try  $x = e^{\lambda t}$ , then  $\dot{x} = \lambda e^{\lambda t}$  and  $\ddot{x} = \lambda^2 e^{\lambda t}$

Substituting to the homogeneous D.E.:

$$\lambda^2 e^{\lambda t} + 3\lambda e^{\lambda t} + 2e^{\lambda t} = 0$$

$$\Leftrightarrow (\lambda^2 + 3\lambda + 2)e^{\lambda t} = 0$$

$$\Leftrightarrow \lambda^2 + 3\lambda + 2 = 0 \quad (\text{since } e^{\lambda t} > 0 \ \forall t)$$

$$\Leftrightarrow (\lambda + 2)(\lambda + 1) = 0$$

$$\begin{cases} \lambda = -2 \\ \lambda = -1 \end{cases}$$

Thus, the general solution for the homogeneous differential equation is:

$$x(t) = A e^{-2t} + B e^{-t} \quad (A, B: \text{const})$$

Find a trivial solution for the differential equation.

$$\text{Since } \sinh(t) = \frac{e^t - e^{-t}}{2}$$

let  $x(t) = ae^t + be^{-t}$  be a solution

However,  $e^{-t}$  is appeared in the general solution of the homogeneous equation

Then, let  $x(t) = ae^t + bte^{-t}$  be a solution

$$\rightarrow \dot{x}(t) = ae^t + be^{-t} - bte^{-t}$$

$$\ddot{x}(t) = ae^t + (bt - 2b)e^{-t}$$

Substituting to the equation:

$$ae^t + (bt - 2b)e^{-t} + 3ae^t + 3be^{-t} - 3bte^{-t} + 2ae^t + 2bte^{-t} = \frac{e^t - e^{-t}}{2} \quad (\forall t)$$

$$\Rightarrow \begin{cases} a + 3a + 2a = \frac{1}{2} \\ bt - 2b + 3b - 3bt + 2bt = -\frac{1}{2} \end{cases} \quad (\text{holds } \forall t)$$

$$\Rightarrow \begin{cases} a = \frac{1}{12} \\ b = -\frac{1}{2} \end{cases}$$

Thus,  $x(t) = \frac{1}{12}e^t - \frac{t}{2}e^{-t}$  is a solution of the D.E

Therefore,  $x(t) = Ae^{-2t} + Be^{-t} + \frac{1}{12}e^t - \frac{t}{2}e^{-t}$  is the general solution of the D.E.

$$\text{ii), } \ddot{x} + 2\dot{x} + 2x = \exp(t)$$

Homogeneous equation  $\ddot{x} + 2\dot{x} + 2x = 0$

Try  $x = e^{\lambda t}$ , then  $\dot{x} = \lambda e^{\lambda t}$ ,  $\ddot{x} = \lambda^2 e^{\lambda t}$

Substituting to the homogeneous equation:

$$(\lambda^2 + 2\lambda + 2)e^{\lambda t} = 0$$

$$\lambda^2 + 2\lambda + 2 = 0 \quad (\text{since } e^{\lambda t} > 0 \ \forall t)$$

$$(\lambda + 1)^2 = -1$$

$$\lambda = -1 \pm i$$

Thus, the general solution for the homogeneous equation is:

$$x(t) = Ae^{-t} \cos t + Be^{-t} \sin t \quad (A, B \in \mathbb{R})$$

Find a trivial solution for the D.E.

Let  $x = ae^t$  be a solution

$$\dot{x} = ae^t, \ddot{x} = ae^t$$

Substituting to the D.E.:

$$ae^t + 2ae^t + 2ae^t = e^t \quad (\forall t)$$

$$5ae^t = e^t \quad (\forall t)$$

$$5a = 1 \quad (\text{since } e^t > 0 \ \forall t, \forall t)$$

$$a = \frac{1}{5}$$

Thus,  $x = \frac{e^t}{5}$  is a solution of the initial D.E.

Therefore,  $x = Ae^{-t} \cos t + Be^{-t} \sin t + \frac{e^t}{5}$  (with  $A, B \in \mathbb{R}$ )

$$\text{iii, } \ddot{x} + 2\dot{x} + x = t^2$$

Homogeneous equation:  $\ddot{x} + 2\dot{x} + x = 0$

Try  $x = e^{\lambda t}$ , then  $\dot{x} = \lambda e^{\lambda t}$ ,  $\ddot{x} = \lambda^2 e^{\lambda t}$

Substituting to the homogeneous equation:

$$\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + e^{\lambda t} = 0$$

$$(\lambda^2 + 2\lambda + 1) e^{\lambda t} = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

(since  $e^{\lambda t} > 0 \forall t$ )

$$\lambda = -1$$

Thus, the general solution for the homogeneous equation is

$$x = A e^{-t} + B t e^{-t} \quad (A, B \in \mathbb{R})$$

Find a solution for the initial equation

$$\text{Try } x = a_2 t^2 + a_1 t + a_0$$

$$\dot{x} = 2a_2 t + a_1$$

$$\ddot{x} = 2a_2$$

Substituting to the D.E:

$$2a_2 + 4a_2 t + 2a_1 + a_2 t^2 + a_1 t + a_0 = t^2$$

$$\rightarrow \begin{cases} 2a_2 + 2a_1 + a_0 = 0 \\ 4a_2 + a_1 = 0 \end{cases}$$

$$a_2 = 1$$

$$a_2 = 1$$

$$a_1 = -4$$

$$a_0 = 6$$

Thus,  $x = t^2 - 4t + 6$  is a solution of the D.E

Therefore,  $x = A e^{-t} + B t e^{-t} + t^2 - 4t + 6$  is the general solution of the D.E (with  $A, B \in \mathbb{R}$ )