

SML

(I)

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Def

Let $M_n(\mathbb{K}[x])$ be the set of n -square matrices that elements are \mathbb{K} -coefficient polynomial.

Def

$A(x) \in M_n(\mathbb{K}[x])$ is regular matrix

$$\Leftrightarrow \exists X(x) \in M_n(\mathbb{K}[x]) \text{ s.t. } A(x)X(x) = X(x)A(x) = E$$

$X(x)$ is called inverse matrix

$$P(i,j) := \begin{pmatrix} 1 & i & j \\ i & 1 & 0 \\ j & 0 & 1 \end{pmatrix}$$

$$Q(i;c) := \begin{pmatrix} 1 & & & \\ & 1 & c & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad c \in \mathbb{K}, c \neq 0$$

$$R(i,j;c(x)) := \begin{pmatrix} 1 & & & \\ & 1 & \dots & c(x) \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad c(x) \in M_n(\mathbb{K}[x])$$

$$\det P(i,j) = \det R(i,j;c(x)) = 1 \in \mathbb{K}$$

$$\det Q(i;c) = c \in \mathbb{K}$$

Def

Elementary transformation that multiply $P(i,j)$, $Q(i;c)$ and $R(i,j;c(x))$ from left (right).

Def.

$A(x), B(x) \in M_n(\mathbb{K}[x])$. If $A(x)$ is transformed to $B(x)$, we call that $A(x)$ and $B(x)$ are equivalent, and write $A(x) \sim B(x)$.

Def

Let $A(x) \in M_n(\mathbb{K}[x])$, $k \in \mathbb{Z}$ ($0 \leq k \leq n$). $(n-k)$ th minor determinant of $A(x)$ is the determinant of $k \times k$ matrix obtained from $A(x)$ deleting $n-k$ rows and $n-k$ columns.

Def

$A(x) \in M_n(\mathbb{K}[x])$, $d_k(x)$ is the determinant divisor of $(n-k)$ th order if d_k is G.C.D of the $(n-k)$ th minor. (The leading coefficient of d_k is equal to 1.) If all minors 0, d_k is equal to 0.

Th. [1.1]

$A(x) \in M_n(\mathbb{K}[x])$ is invertible matrix. $\Leftrightarrow \det A(x) \in \mathbb{K}$, $\det A(x) \neq 0$.

pf

$$\Rightarrow: \det A(x) \det A(x)^{-1} = \det A(x) \tilde{A}(x) = \det E = 1$$

The degree of right hand side is 0. $\det A(x)$ and $\det A(x)^{-1}$ are polynomial of x , so $\det A(x) \in \mathbb{K}$, $\det A(x) \neq 0$.

\Leftarrow :

$\det A(x) \in \mathbb{K}$, $\det A(x) \neq 0$. Let $\tilde{A}(x)$ is adjugate matrix of $A(x)$.

$$\tilde{A}(x) A(x) = A(x) \tilde{A}(x) = E - \det A(x)$$

$$\text{Therefore, } \frac{1}{\det A(x)} \tilde{A}(x) = \tilde{A}(x)$$



Th. [1.3]

Let $d_k^A(x)$ be the determinant divisor of $A(x)$ and $d_k^B(x)$ be the determinant divisor of $B(x)$.

$$A(x), B(x) \in M_n(\mathbb{K}[x]), A(x) \sim B(x) \Rightarrow d_k^A(x) = d_k^B(x) \text{ for all } k.$$

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pf.

We have to say the unchangeable of the determinant divisor after elementary transformation.

It's obvious that the determinant divisor of $A_{(i)}^*$, $P_{(i;j)} A_{(i)}$, $A_{(i)} P_{(i;j)}$, $Q_{(i;j;c)} A_{(i)}$, $A_{(i)} Q_{(i;j;c)}$ are same.

$A'_{(i)} := R_{(i;j;c_{(i)})} A_{(i)}$. Let $d_{k(i)}$ be the determinant divisor of $A'_{(i)}$.

$d_{k(i)} = d_{k(i)}$ if they don't i th row, or they include i th row and j th row. Former is obvious, so proof the latter.
Let p be the first row of submatrix and g be the last one.
And let Δ be the determinant of $A_{(i)}$, Δ' be the determinant of corresponding submatrix of $A'_{(i)}$

$$\Delta' = \sum_{\tau \in G_n} \text{sgn}(\tau) \cdot A_{p\tau(p)} \cdots (A_{i\tau(i)} + c_{(i)} A_{j\tau(i)}) \cdots A_{j\tau(j)} \cdots A_{g\tau(g)}$$

$$= \sum_{\tau \in G_n} \text{sgn}(\tau) A_{p\tau(p)} \cdots A_{i\tau(i)} \cdots A_{j\tau(j)} \cdots A_{g\tau(g)}$$

$$+ c_{(i)} \sum_{\tau \in G_n} \text{sgn}(\tau) A_{p\tau(p)} \cdots A_{i\tau(i)} A_{j\tau(i)} A_{j\tau(j)} \cdots A_{g\tau(g)}$$

$$= \sum_{\tau \in G_n} \text{sgn}(\tau) A_{p\tau(p)} \cdots A_{i\tau(i)} \cdots A_{j\tau(j)} \cdots A_{g\tau(g)} = \Delta$$

Let submatrix include i th row but j th row.

Let $A_{(i;j)}$ be matrix that exchange i th row and j th row of $A_{(i)}$, and let Δ_1 be the determinant of submatrix of $A_{(i;j)}$. Then, submatrix doesn't include j th row, so $\Delta' = \Delta + c_{(i)} \Delta_1$.

Δ and Δ_1 are the minor of $A_{(i)}$, so Δ and Δ_1 can be divided by $d_{k(i)}$. Therefore, Δ' can be divided by $d_{k(i)}$, so $d_{k(i)}$ can be divided by $d_{k(i)}$. Elementary transformation is invertible transformation, it's obvious that $d_{k(i)}$ can be divided by $d_{k(i)}$.

The leading factor of $d_{k(i)}$ and $d_{k(i)}$ are 1, so $d_{k(i)} = d_{k(i)}$.

Th. [1.2]

$\forall A_{12n} \in M_n(K[x])$, $\exists! r, e_i(x)$ ($i, r \in \mathbb{Z}_+, 0 \leq r < n, 0 < i \leq n$)

s.t.

$$A_{12n} \sim \begin{pmatrix} e_1(x) & & & \\ & e_2(x) & & \\ & & \ddots & \\ & & & e_r(x) \\ & & & & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix} \quad \text{... } \otimes$$

- 1) The leading factor of $e_i(x)$ is 1
- 2) $e_i(x)$ can be divided by $e_{i+1}(x)$

→ r is called rank, $e_i(x)$ is called elementary divisor, and RHS of \otimes is called canonical form.

prf
n=1.

Obvious.

Suppose that $n-1$ square matrix fulfill Th. [1.2].

If $A_{120} = 0$, A_{121} is already canonical form, so assume $A_{120} \neq 0$.
 Think the set S ; include whole matrix that equivalent to A_{120} and
 (1, 1) element is not 0. Let $B_{120} \in S$ be the matrix that (1, 1)
 polynomial is the least degree.

$$A_{121} \sim B_{120} = \begin{pmatrix} e_1(x) & b_{12}(x) & \cdots & b_{1n}(x) \\ b_{21}(x) & b_{22}(x) & & \\ \vdots & \vdots & \ddots & | \\ b_{n1}(x) & \cdots & \cdots & b_{nn}(x) \end{pmatrix}$$

All element in 1st row and 1st column can be divided by $e_1(x)$
 because if $b_{ij}(x)$ can not be divided by $e_1(x)$, we can take
 polynomial $r_{ij}(x)$ that lower degree than $e_1(x)$ such that
 $b_{ij}(x) = e_1(x) q_{ij}(x) + r_{ij}(x)$.

Therefore

$$C_{121} = \begin{pmatrix} e_1(x) & 0 & \cdots & 0 \\ 0 & b_{22}(x) & \cdots & b_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2}(x) & \cdots & b_{nn}(x) \end{pmatrix} \sim A_{121}.$$

Because of supposition,

$$\begin{pmatrix} b_{22}(x) & \cdots & b_{2n}(x) \\ \vdots & \ddots & \vdots \\ b_{n2}(x) & \cdots & b_{nn}(x) \end{pmatrix} \sim \begin{pmatrix} e_2(x) & & & \\ & e_3(x) & & \\ & & \ddots & \\ & & & e_n(x) \\ & & & & 0 & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix}$$

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$e_{1(x)}$ can be divisible by $e_{1(x)}$. Because if $e_{1(x)}$ can not be divisible by $e_{1(x)}$, there exist $r_{1(x)}$ such that $e_{1(x)} = e_{1(x)}p_{1(x)} + r_{1(x)}$. And degree of $r_{1(x)}$ is lower than $e_{1(x)}$.

$$\text{Therefore, } A(x) \sim \begin{pmatrix} e_{1(x)} & & \\ & \ddots & \\ & & e_{r(x)} \\ & & & \ddots & 0 \\ & & & & 0 \end{pmatrix}$$

Next prf is the uniqueness of canonical form. Let $D(x)$ be the canonical form of $A(x)$ and the determinant divisor of $D(x)$ be $d_{1(x)}, \dots, d_{r(x)}$. It is clear that.

$$d_k(x) = \begin{cases} e_{1(x)} \cdots e_{k(x)} & \text{for } k \leq r. \\ 0 & \text{for } k > r \end{cases}$$

Therefore, $e_{k(x)} = d_k(x) / d_{k-1}(x)$.

$\therefore e_{k(x)}$ and r exist uniquely for any $A(x) \in M_n(\mathbb{K}[x])$



Def

$A \in M_n(\mathbb{K})$. $xE - A$ is called characteristic x -matrix

Def

$A(x) \in M_n(\mathbb{K}[x])$. For any $A(x)$, we can express this matrix like this:

$$A(x) = A_0 x^k + \cdots + A_k$$

k is called power degree of $A(x)$, $\deg A(x) := k$

Cor [1.4]

$A(x), B(x) \in M_n(\mathbb{K}[x])$.

$A(x) \sim B(x) \iff$ The rank and elementary divisors of $A(x)$ and $B(x)$ are same.

prf

\sim is equivalence relation, so this corollary is obvious.



Cor [1.5]

- 1) Invertible α -matrix is equivalent to identity. Inverse is also true.
- 2) All invertible α -matrix can be written by the product of elementary matrix

pf

1).

Let $P_{1(2)}$ and $Q_{1(2)}$ be a product of elementary matrix, $D_{1(2)}$ be a canonical form of $A_{1(2)}$. $A_{1(2)}$ is invertible matrix

If $A_{1(2)} \sim E$, there exist $P_{1(2)}, Q_{1(2)}$ such that

$$D_{1(2)} = P_{1(2)} A_{1(2)} Q_{1(2)}.$$

$$\det D_{1(2)} = \det(P_{1(2)} A_{1(2)} Q_{1(2)})$$

$$= \det P_{1(2)} \det A_{1(2)} \det Q_{1(2)}.$$

$$\det P_{1(2)}, \det Q_{1(2)}, \det A_{1(2)} \in \mathbb{K} \quad (\because \text{Th. [1.1]}).$$

$$\therefore \det D_{1(2)} \in \mathbb{K}.$$

$D_{1(2)}$ is canonical form, so diagonal elements of $D_{1(2)}$ are 1. Therefore, invertible α -matrix is equivalent to identity. Inverse is obvious.

2)

If $A_{1(2)}$ is invertible matrix, $A_{1(2)} \sim E$. And there exist elementary matrix $P_{1(2)} \dots P_{k(2)}$; $Q_{1(2)} \dots Q_{l(2)}$ such that

$$P_{1(2)} \dots P_{k(2)} A_{1(2)} Q_{1(2)} \dots Q_{l(2)} = E$$

$$A_{1(2)} = P_{k(2)}^{-1} \dots P_{1(2)}^{-1} Q_{l(2)}^{-1} \dots Q_{1(2)}^{-1}$$

And $P_{j(2)}^{-1}$ and $Q_{j(2)}^{-1}$ are also elementary matrix. ■

Cor [1.6]

$A_{1(2)} \sim B_{1(2)} \Leftrightarrow \exists P_{1(2)}, Q_{1(2)}$ (invertible matrix). s.t.

$$B_{1(2)} = P_{1(2)} A_{1(2)} Q_{1(2)}$$

pf

Obvious from Cor [1.5] 2). ■

SML IV

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Th. [1.7]Let $A(x)$ and $B(x)$ be like

$$A(x) = A_0 x^k + \dots + A_k \quad A_0 \neq 0$$

$$B(x) = B_0 x^l + \dots + B_l \quad B_0 \neq 0$$

If B_0 is regular, $Q_1(x)$ and $R_1(x)$ uniquely exist such that

$$A(x) = B(x) Q_1(x) + R_1(x), \quad (R_1(x) = 0 \text{ or } \deg R_1(x) > \deg B(x))$$

 $Q_2(x)$ and $R_2(x)$ uniquely exist such that

$$A(x) = Q_2(x) B(x) + R_2(x) \quad (R_2(x) = 0 \text{ or } \deg B(x) > \deg R_2(x))$$

prf.

If $k=0, l>0$,

$$Q_1(x) = 0, R_1(x) = A(x)$$

If $l=0$,

$$B(x) = B_0, Q_1(x) = B_0^{-1} A(x), R_1(x) = 0$$

Suppose that Th. [1.7] is true for all $X(x) \in M_n(\mathbb{K}[x])$, $\deg X(x) < k$.If $k < l$.

$$Q_1(x) = 0, R_1(x) = A(x)$$

If $k \geq l$ Let $A'(x)$ be $A'(x) \in M_n(\mathbb{K}[x])$, $A'(x) = A(x) - B(x) B_0^{-1} A_0 x^l$

$$A'(x) = (A_0 - B_1 B_0^{-1} A_0) x^{k-l} + \dots + A_k$$

$$\therefore \deg A'(x) = k-1$$

Therefore, $\exists! Q_1'(x), R_1(x)$ ($\deg R_1(x) < l$) s.t.

$$A'(x) = B(x) Q_1'(x) + R_1(x)$$

Then

$$A(x) = A'(x) + B(x) B_0^{-1} A_0 x^{k-l}$$

$$= B(x) (B_0^{-1} A_0 x^{k-l} + Q_1'(x)) + R_1(x)$$

$$= B(x) Q_1(x) + R_1(x) \quad Q_1(x) := B_0^{-1} A_0 x^{k-l} + Q_1'(x)$$

If $Q_1'(x)$ and $R_1'(x)$ fulfill the condition,

$$B(x) \{Q_1(x) - Q_1'(x)\} = R_1(x) - R_1'(x)$$

$$\deg [B_{11} \{ Q_{1100} - Q_1'(x) \}] > \deg [R_{1100} - R_1'(x)]$$

$$\therefore Q_{1100} = Q_1' \\ R_{1100} = R_1'$$

Therefore, Q_{1100} and R_{1100} are unique.

Th. [1.8]

$A, B \in M_n(\mathbb{K})$, A and B are similar

$\Leftrightarrow \alpha E - A$ and $\alpha E - B$ are equivalent

prf

\Rightarrow :

If A and B are similar, there exist P ($\det P \neq 0$) such that $B = P^{-1}AP$.

$$\begin{aligned}\alpha E - B &= \alpha P^{-1}P - P^{-1}AP \\ &= P^{-1}(\alpha E - A)P\end{aligned}$$

Therefore, $\alpha E - A$ and $\alpha E - B$ are equivalent.

(\because Cor [1.6])

\Leftarrow :

If $\alpha E - A$ and $\alpha E - B$ are equivalent, there exist invertible matrices such that $(\alpha E - A)P(x) = Q_{1100}(\alpha E - B)$

$$\therefore \text{Th. [1.7]} \quad P_{1100} = P_{1100}(\alpha E - B) + P$$

$$Q(x) = (\alpha E - A)Q_{1100} + Q \quad (\because E \text{ is regular.})$$

$$\deg P = \deg Q = 0 \text{ because } \deg (\alpha E - A) = \deg (\alpha E - B) = 1$$

$$\therefore 0 = (\alpha E - A)P(x) - Q_{1100}(\alpha E - B)$$

$$= (\alpha E - A)P_{1100}(\alpha E - B) + (\alpha E - A)P$$

$$- (\alpha E - A)Q_{1100}(\alpha E - B) - Q(\alpha E - B)$$

$$= (\alpha E - A)(P_{1100} - Q_{1100})(\alpha E - B) + (\alpha E - A)P - Q(\alpha E - B)$$

$$(\alpha E - A)(P_{1100} - Q_{1100})(\alpha E - B) = Q(\alpha E - B) - (\alpha E - A)P$$

... *

If $P_{1100} = Q_{1100}$,

$$Q(\alpha E - B) = (\alpha E - A)P$$

$$(Q - P)\alpha E = QB - AP$$

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If $(LHS) \neq 0$, $(RHS) \neq 0$, $\deg(LHS) = 1$, $\deg(RHS) = 0$
 So $LHS \neq RHS$. Therefore $(LHS) = (RHS) = 0$

$$\therefore Q = P, PB = AP$$

The prf of regularity of P is latter.

If $P_{1(x)} \neq Q_{1(x)}$

$\deg(LHS \oplus) \geq 2$, $\deg(RHS \oplus) \leq 1$, so $(LHS \oplus) \neq (RHS \oplus)$
 Therefore, $P_{1(x)} = Q_{1(x)}$

The next prf is the regularity of P .

$P_{1(x)}$ is invertible matrix, and there exist $R_{1(x)} \in M_n(K[x])$
 and $R \in M_n(K)$ such that

$$P_{1(x)}^{-1} = R_{1(x)}(xE - A) + R$$

$$E = P_{1(x)}^{-1} P_{1(x)}$$

$$= [R_{1(x)}(xE - A) + R][P_{1(x)}(xE - B) + P]$$

$$= [R_{1(x)}(xE - A) + R]P_{1(x)}(xE - B) + R_{1(x)}(xE - A)P + RP$$

$$= [R_{1(x)}(xE - A) + R]P_{1(x)}(xE - B) + R_{1(x)}Q(xE - B) + RP$$

$$= S(x)(xE - B) + RP$$

$$S(x) := [R_{1(x)}(xE - A) + R]P_{1(x)} + R_{1(x)}Q$$

$$\deg(RHS) = 0 \quad \therefore \deg(LHS) = 0$$

$$\deg(xE - B) = 1, \quad xE - B = 0, \text{ so } S(x) = 0.$$

$$\therefore E = RP$$



According to prf, $P_{1(x)}, Q_{1(x)}, P_{1(x)} \in M_n(K[x])$, $P \in M_n(K)$
 such that

$$(xE - A)P_{1(x)} = Q_{1(x)}(xE - B)$$

$$B = P_{1(x)}^{-1}AP$$

$$P(x) = P_{1(x)}(xE - B) + P,$$

there hold equations like

$$P(x) = P_{1(x)}(xE - B) + P$$

$$Q(x) = (xE - Q)P_{1(x)} + P$$

Def

$$A(x) \in M_m(\mathbb{K}[x]), B(x) \in M_n(\mathbb{K}[x])$$

$$A(x) + B(x) := \begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix}$$

Def

$$J(a, b) := \begin{pmatrix} a & & & 0 \\ & a & & \\ & & a & \\ 0 & & & a \end{pmatrix}, J(a, b) \in M_b(\mathbb{K})$$

$J(a, b)$ is called Jordan cell.

$$J := J(a_1, b_1) + J(a_2, b_2) + \dots + J(a_s, b_s)$$

J is called Jordan matrix

Appendix Th. [1.4]

Let $f(x)$ and $g(x)$ be polynomials and $d(x)$ be G.C.D of $f(x)$ and $g(x)$. $f(x), g(x), d(x) \in \mathbb{K}[x]$
 $\exists u(x), v(x) \in \mathbb{K}[x]$ such that $f(x)u(x) + g(x)v(x) = d(x)$

prf

$$A := \{F(x) | F(x) = f(x)p(x) + g(x)q(x), F(x), p(x), q(x) \in \mathbb{K}[x]\}$$

Let $\psi(x) = u(x)f(x) + v(x)g(x) \in A$ be a polynomial that degree is minimum. Let $d(x)$ be a G.C.D. of $f(x)$ and $g(x)$.

$$\begin{aligned} f(x) &= \psi(x)g(x) + r_1(x) & \text{for } \deg r_i < \deg \psi(x) \text{ or } r_i = 0 \\ g(x) &= \psi(x)g(x) + r_2(x). & i=1,2 \end{aligned}$$

$$\begin{aligned} r_1(x) &= f(x) - \psi(x)g(x) \\ &= f(x) - (u(x)f(x) + v(x)g(x))g(x) \\ &= (1 - u(x)g(x))f(x) - v(x)g(x) \end{aligned}$$

$\therefore r_1(x) \in A$. Likewise, $r_2(x) \in A$.

According to definition of $\psi(x)$, $r_1(x) = r_2(x) = 0$, so $\psi(x)$ is a common divisor of $f(x)$ and $g(x)$.

$\therefore d(x)$ is a G.C.D., so $d(x)$ can be divided by $\psi(x)$.

Meanwhile, $\psi(x) \in A$. Therefore, $\psi(x)$ can be divided by $d(x)$.

$$\therefore \psi(x) = c d(x), (c \in \mathbb{K}, c \neq 0)$$

SML (VI)

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Th. [2.1]

$$A(x) := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & f(x) \end{pmatrix} \quad B(x) := \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & g(x) \end{pmatrix}$$

(1) If $g(x)$ can be divided by $f(x)$,

$$A(x) + B(x) \sim B(x) + A(x) \sim \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & f(x), g(x) \end{pmatrix}$$

(2) If $\gcd(f(x), g(x)) = 1$.

$$A(x) + B(x) \sim \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & f(x)g(x) \end{pmatrix}$$

prf.

(1) : Obviously, this fulfill Th. [1.2] ..

(2) : Because of Appendix [1.4], $\exists u(x), v(x) \in K[x]$ such that $f(x)u(x) + g(x)v(x) = 1$.

$$\begin{pmatrix} 1 & \\ & f(x)g(x) \end{pmatrix} = \begin{pmatrix} 1 & v(x) \\ -g(x) & f(x)u(x) \end{pmatrix} \begin{pmatrix} f(x) & 0 \\ 0 & g(x) \end{pmatrix} \begin{pmatrix} u(x) & -g(x)v(x) \\ 1 & f(x) \end{pmatrix},$$

$$\det \begin{pmatrix} 1 & v(x) \\ -g(x) & f(x)u(x) \end{pmatrix} = \det \begin{pmatrix} u(x) & -g(x)v(x) \\ 1 & f(x) \end{pmatrix} = 1$$

Therefore $\begin{pmatrix} 1 & v(x) \\ -g(x) & f(x)u(x) \end{pmatrix}$ and $\begin{pmatrix} u(x) & -g(x)v(x) \\ 1 & f(x) \end{pmatrix}$ are invertible matrix.Because of Cor [1.6], $\begin{pmatrix} f(x) & 0 \\ 0 & g(x) \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & f(x)g(x) \end{pmatrix}$

Def.

$$xE - J(d, k) = \begin{pmatrix} x-\alpha & 1 & & & 0 \\ & x-\alpha & \ddots & & \\ & & \ddots & \ddots & 0 \\ 0 & & & x-\alpha & \end{pmatrix}$$

This matrix is called characteristic Jordan matrix.

Def.

$J := J(\alpha_1, k_1) + J(\alpha_2, k_2) + \dots + J(\alpha_s, k_s)$. $k_1 + k_2 + \dots + k_s = n$.

J is called n th order Jordan matrix.

To make easy, we change the field \mathbb{K} to \mathbb{C} .

Th. [2.2] ; Goal

$\forall A \in M_n(\mathbb{C})$, $\exists ! J$ (except for how to list Jordan cell)
s.t. A and J are similar.

pf

The keystone of this pf is Th. [1.8].

Let $\alpha_1, \dots, \alpha_p$ be eigenvalues of J .

Each eigenvalue, we take the matrix that degree is maximum.
Then, make the direct sum of these matrices, K_1 .

Because of $\lambda E - J(\alpha, k) \sim \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \lambda - \alpha \end{pmatrix}^k$ and Th. [2.1] (D),

The canonical form of K_1 is

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & e_h(\lambda) \end{pmatrix}$$

Each eigenvalues, we take matrixes that degree is secondary maximum. If the duplication of the eigenvalue is 1, then we don't take that matrix. We take the direct sum of these matrixes, K_2 . Same as K_1 , the canonical form of K_2 is

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & e_{n-1}(\lambda) \end{pmatrix}$$

Obviously, $e_n(\lambda)$ can be divided by $e_{n-1}(\lambda)$.

We continue it until all characteristic Jordan cells are exhausted. Then we gain λ -matrix $K_1(\lambda), K_2(\lambda), \dots, K_r(\lambda)$.

ku baki yny

$$k_i(x) \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & e_{n-i+1}(x) \end{pmatrix}$$

Therefore, because of [2.1],

$$\alpha E - J \sim \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & e_{n+r}(x) \\ & & & \ddots \\ & & & e_n(x) \end{pmatrix}$$

By the above and Th. [1.8], if 2 Jordan matrixes are similar they are same matrixes except for how to rearrange Jordan cell.

On the other hand, if given $e_1(x) \cdots e_n(x)$, ($e_i(x) \neq 0, i=1, \dots, n$). The leading coefficient of $e_i(x)$ is 1. $e_i(x)$ can be divided by $e_i(x)$. There exists Jordan matrix. (uniquely except for how to list Jordan cell.)

Then, $\forall A \in M_n(\mathbb{C})$, $d_n(x) = \det(\alpha E - A) \neq 0$. So, there exist elementary divisor $e_1(x), \dots, e_n(x)$. Therefore, this canonical form is equivalent to Characteristic Jordan matrix

$$\alpha E - A \sim \begin{pmatrix} e_1(x) & & & \\ & e_2(x) & & \\ & & \ddots & \\ & & & e_n(x) \end{pmatrix} \sim \alpha E - J$$

Because of Th. [1.8], $\forall A \in M_n(\mathbb{C}) \exists ! J$ (except for how to rearrange Jordan cells). such that A and J are similar.

Reference.

線型代數入門, 斎藤正彦, 総合大学出版会
 (Introduction to linear algebra, Saito Masahiko,
 Tokyo Univ. publisher.)

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Date

