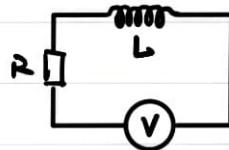


# The solution of electric circuit.

## ① LR circuit.

According to the Krichhoff's laws

$$L \frac{dI(t)}{dt} + R I(t) = V(t). \quad \textcircled{*}$$



i) Let  $V(t) = V_0$  with the initial condition  $I(t=0) = 0$  ( $\dot{I}(t=0) = \frac{V_0}{R}$ )

we can solve this problem :

Notice that the solution of

$$L \frac{dI(t)}{dt} + R I(t) = 0$$

is that.

$$I(t) = A e^{-\frac{R}{L}t}.$$

Here  $A$  is constant that depends on the initial condition.

Then, we let the solution of  $\textcircled{*}$  can be write in a form of

$$I(t) = A(t) e^{-\frac{R}{L}t}$$

Then, it satisfies

$$L \frac{d}{dt}(A(t) e^{-\frac{R}{L}t}) + R(A(t) e^{-\frac{R}{L}t}) = V_0$$

$$\Rightarrow L \frac{d}{dt}(A(t)) e^{-\frac{R}{L}t} = V_0$$

$$\begin{aligned} \text{Then } A(t) &= \text{Const.} + \int_0^t \frac{V_0}{L} e^{\frac{R}{L}s} ds \\ &= \text{Const.} + \frac{V_0}{R} e^{\frac{R}{L}t} \end{aligned}$$

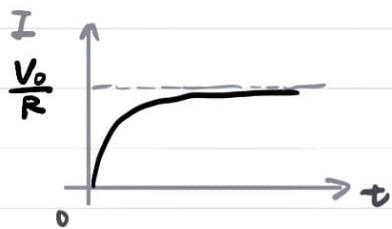
And the solution can be written in a form of

$$I(t) = (\text{Const.} + \frac{V_0}{R} e^{\frac{R}{L}t}) e^{-\frac{R}{L}t}$$

Using the initial condition, we can get

$$I(t) = \frac{V_0}{R} (1 - e^{-\frac{R}{L}t}).$$

Then sketch it in a graph :



ij. Let  $V(t) = V_0 \cos \omega t$  and  $R = 0$  with the initial condition  $I(0) = 0$ .

$$\text{Then, } L \frac{dI}{dt} = V_0 \cos \omega t$$

$$I(t) = \frac{V_0}{\omega L} \sin \omega t = \frac{V_0}{\omega L} \cos(\omega t - \frac{\pi}{2})$$

It means that  $I(t)$  is vibrate with the same frequency of  $V(t)$  and have  $\frac{\pi}{2}$  phase difference with  $V(t)$ .

Using the complex voltage and complex current, we can get

$$V(\omega) = V_0 e^{i\omega t}$$

$$I(\omega) = \frac{V_0}{\omega L} e^{i\omega t} \cdot e^{-\frac{2\pi}{\omega} i} = \frac{i}{i} = -i$$

$$= \frac{V_0}{i\omega L} e^{i\omega t} \quad (L \frac{dI}{dt} = i\omega L \frac{V_0}{i\omega L} e^{i\omega t} = V_0 e^{i\omega t})$$

So we can define a  $Z_L \in \mathbb{C}$  and.

$$Z_L = \frac{V(\omega)}{I(\omega)} = i\omega L$$

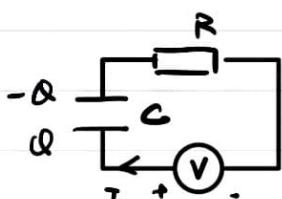
## ②. CR circuit.

According to the Kirchhoff's laws

$$\frac{Q(t)}{C} + I(t)R = V(t)$$

Notice the connection between  $Q(t)$  and  $I(t)$

$$I(t) = \frac{d}{dt} Q(t)$$



Then the equation of  $Q(\omega)$  is.

$$R \frac{dQ(\omega)}{d\omega} + \frac{1}{C} Q(\omega) = V(\omega).$$

ii). Let  $V(\omega) = V_0$ , with the initial condition  $Q(0) = 0$ .

We can get the solution in a same form of ① ii,

$$\begin{aligned} \Rightarrow Q(\omega) &= \left[ \text{Const} + \int_0^\omega \frac{V_0}{R} e^{\frac{1}{CR}t} dt \right] e^{-\frac{1}{CR}\omega} \\ &= CV_0 - \text{Const} \cdot e^{-\frac{1}{CR}\omega} \end{aligned}$$

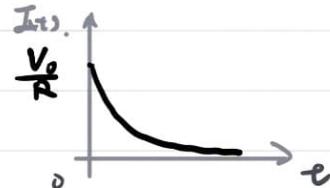
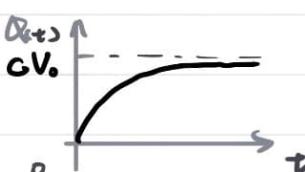
Using the initial condition,  $Q(0) = 0$ , we can get

$$Q(\omega) = CV_0 (1 - e^{-\frac{1}{CR}\omega})$$

And

$$I(\omega) \cdot \frac{dQ(\omega)}{d\omega} = \frac{V_0}{R} e^{-\frac{1}{CR}\omega}.$$

Sketching  $Q(\omega)$  and  $I(\omega)$  in graph:



iii). Let  $V = V_0 \cos \omega t$  and  $R = 0$

$$\frac{1}{C} Q(\omega) = V_0 \cos \omega t$$

Then

$$Q(t) = CV_0 \cos \omega t$$

And

$$I(t) \cdot \frac{dQ(t)}{dt} = -\omega C V_0 \sin \omega t = \omega C V_0 \cos \left( \omega t + \frac{\pi}{2} \right).$$

Write it in complex space, we can get

$$V(\omega) = V_0 e^{i\omega t}$$

$$I(t) = \frac{d}{dt} (CV(t)) = i\omega C V_0 e^{i\omega t} = \omega C V_0 e^{i(\omega t + \frac{\pi}{2})}$$

So we can also define a resistance  $Z_C$

$$Z_C = \frac{V_{C0}}{I_{C0}} = \frac{1}{i\omega C}$$

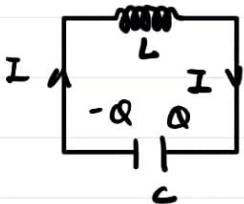
### ③. LC circuit.

According to the Kirchhoff's laws

$$L \frac{dI}{dt} + \frac{Q}{C} = 0.$$

Notice that  $I = \frac{dQ}{dt}$

$$\text{Then } L \frac{d^2Q}{dt^2} + \frac{1}{C} Q_{00} = 0. \quad (\frac{d^2Q}{dt^2} + \frac{1}{CL} Q_{00} = 0)$$



To solve the problem, let  $I_{C0} = A e^{i\lambda t}$  ( $A \neq 0$ )

Then,

$$(i\lambda^2 + \frac{1}{CL}) A e^{i\lambda t} = 0 \Rightarrow \lambda = \pm \frac{1}{\sqrt{CL}}$$

$$\text{So } I_{C0} = A e^{-i\omega t} + B e^{i\omega t} \quad (A, B \in \mathbb{C}, \omega = \frac{1}{\sqrt{CL}}).$$

$$= A(\cos \omega t - i \sin \omega t) + B(\cos \omega t + i \sin \omega t)$$

$$= (A+B) \cos \omega t + i(B-A) \sin \omega t.$$

Notice that  $I(t) \in \mathbb{R}$ . So  $B = A^*$  and

$$I_{C0} = C_1 \cos \omega t + C_2 \sin \omega t$$

$C_1, C_2$  can be solved from the initial condition.

For example, let  $I(t=0) = 0$  and  $Q(t=0) = -CV_0$

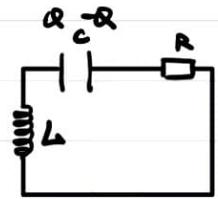
$$\begin{cases} \left. \frac{dI}{dt} \right|_{t=0} = \omega C_2 = -\frac{1}{CL} Q_{00} = -\frac{V_0}{L} \\ I(t=0) = C_1 = 0 \end{cases}$$

$$\Rightarrow I_{C0} = \frac{V_0}{\omega L} \sin \omega t \quad \text{for } \omega = \frac{1}{\sqrt{LC}}$$

## ④. LCR circuit without external power source.

According to the Kichhoff's laws

$$\frac{Q(\omega)}{C} + L \frac{dI(\omega)}{dt} + RI(\omega) = 0$$



Notice that  $\frac{d}{dt}Q(\omega) = I(\omega)$ . we can get

$$L \frac{d^2I(\omega)}{dt^2} + R \frac{dI(\omega)}{dt} + \frac{1}{C} I(\omega) = 0$$

$$\Rightarrow \frac{d^2I}{dt^2} + 2\eta \frac{dI}{dt} + \omega_0^2 I(\omega) = 0 \quad (\eta = \frac{R}{2L}, \omega_0^2 = \frac{1}{LC})$$

Here, let the initial condition:  $\begin{cases} I(t=0) = 0 \\ Q(t=0) = -CV_0 \end{cases}$

To solve the problem, let  $I(\omega) = C e^{\lambda \omega}$  and we can get a equation of A:

$$\lambda^2 + 2\eta \lambda + \omega_0^2 = 0$$

Consider  $D = \eta^2 - \omega_0^2$ , and the solution of this equation is  $\lambda_1 = -\eta + \sqrt{D}$ ,  $\lambda_2 = -\eta - \sqrt{D}$ .

$$i). \eta = \omega_0 \quad (R^2 = \frac{4L}{C})$$

$$\text{for } D = \eta^2 - \omega_0^2 = 0 \quad , \quad \lambda_1 = \lambda_2 = \lambda_0 = -\eta$$

Then, rewrite  $\frac{d^2I}{dt^2} + 2\eta \frac{dI}{dt} + \omega_0^2 I(\omega) = 0$ :

$$\left( \frac{d^2}{dt^2} + 2\eta \frac{d}{dt} + \omega_0^2 \right) I = (\frac{d}{dt} - \lambda_0)^2 I = 0$$

Using the equation:

$$(\frac{d}{dt} - \lambda_0) f = e^{\lambda_0 t} \frac{d}{dt} (e^{-\lambda_0 t} f)$$

we can get

$$e^{\lambda_0 t} \frac{d}{dt} \left[ e^{\lambda_0 t} \left\{ e^{-\lambda_0 t} \frac{d}{dt} (e^{-\lambda_0 t} I) \right\} \right] = 0$$

So

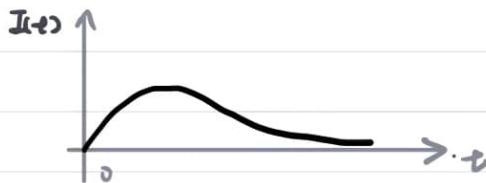
$$\frac{d^2}{dt^2} (e^{-\lambda_0 t} I) = 0 \Rightarrow I = (A + Bt) e^{\lambda_0 t} \quad (A, B = \text{const.})$$

Using the initial condition.

$$\Rightarrow I(0) = A = 0$$
$$I(t)|_{t=0} = [(A + Bt)\lambda_0 + B] e^{\lambda_0 t}|_{t=0} = B = \frac{V_0}{L}$$

Then,  $I(t) = \frac{V_0}{L} t e^{-\lambda_0 t} = \frac{V_0}{L} t e^{-\frac{R}{2L} t}$

And we can sketch it :



ii).  $D > 0 \quad (\eta^2 > \omega_0^2)$ .

$$\lambda_1 = -\eta + \sqrt{D} = -\eta + \sqrt{\eta^2 - \omega_0^2}$$

$$\lambda_2 = -\eta - \sqrt{D} = -\eta - \sqrt{\eta^2 - \omega_0^2}$$

Then,  $I(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \quad (C_1, C_2 \text{ are const.})$

$$= e^{-\eta t} (C_1 e^{\sqrt{\eta^2 - \omega_0^2} t} + C_2 e^{-\sqrt{\eta^2 - \omega_0^2} t})$$

Using the initial condition :

$$I(0) = C_1 + C_2 = 0$$

$$I'(0) = -\eta(C_1 + C_2) + \sqrt{\eta^2 - \omega_0^2}(C_1 - C_2) = \frac{V_0}{2L}$$

Then

$$C_1 = \frac{V_0}{2L} \frac{1}{\sqrt{\eta^2 - \omega_0^2}} = \frac{V_0}{\sqrt{R^2 - \frac{4L^2}{R^2}}} = -C_2$$

$$\Rightarrow I(t) = e^{-\frac{\eta}{2} t} \frac{V_0}{\sqrt{R^2 - \frac{4L^2}{R^2}}} (e^{\sqrt{\frac{R^2}{4L^2} - \frac{1}{4}} t} - e^{-\sqrt{\frac{R^2}{4L^2} - \frac{1}{4}} t})$$
$$= e^{-\frac{\eta}{2} t} \frac{2V_0}{\sqrt{R^2 - \frac{4L^2}{R^2}}} \sinh \sqrt{\frac{R^2}{4L^2} - \frac{1}{4}} t$$

Particularly, when  $\eta^2 \gg \omega_0^2$  ( $\frac{R^2}{4L^2} \gg \frac{1}{Lc}$ )

$$\lambda_1 = -\eta + \sqrt{\eta^2 - \omega_0^2} = -\eta + \eta (1 - \frac{\omega_0^2}{\eta^2})^{\frac{1}{2}} \approx -\frac{\omega_0^2}{2\eta} = -\frac{1}{CR}$$

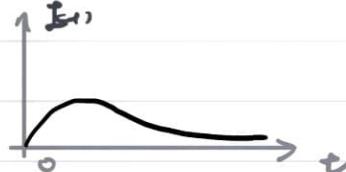
$$\lambda_2 = -\eta - \sqrt{\eta^2 - \omega_0^2} \approx -\eta - \eta (1 - \frac{\omega_0^2}{\eta^2}) \approx -2\eta = -\frac{2}{L}$$

Then

$$C_1 = \frac{V_0}{2L} \frac{1}{\sqrt{\eta^2 - \omega_0^2}} \approx \frac{V_0}{2L \cdot \eta} = \frac{V_0}{RL} = -C_2$$

$$I_{(t)} \approx \frac{V_0}{R} (e^{-\frac{1}{RL}t} - e^{-\frac{2}{L}t})$$

We can sketch it :



iii).  $D < 0$ . ( $\eta^2 < \omega_0^2$ )

$$\text{Set } w = \sqrt{\omega_0^2 - \eta^2}$$

Then

$$\lambda_1 = -\eta + iw$$

$$\lambda_2 = -\eta - iw$$

And

$$I(w) = C_1 e^{\lambda_1 w} + C_2 e^{\lambda_2 w}$$

$$= e^{-\eta w} (C_1 e^{iw} + C_2 e^{-iw}) \quad (C_1, C_2 \text{ are const.})$$

$$= e^{-\eta w} (A \cos wt + B \sin wt) \quad (A, B \text{ are const.})$$

Using the initial condition, we can get

$$I(w) = A = 0$$

$$I'(w) = -\frac{R}{2L} A + wB = \frac{V_0}{L}$$

Then

$$I(w) = \frac{V_0}{wL} e^{-\eta w} B \sin wt$$

$$= \frac{V_0}{\sqrt{\frac{R^2}{4L^2} - \frac{R^2}{L^2}}} e^{-\frac{R}{2L}t} \sin \sqrt{\frac{1}{Lc} - \frac{R^2}{L^2}} t$$

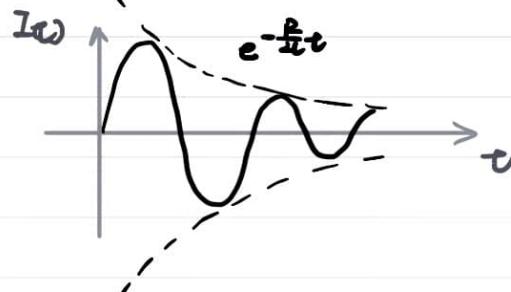
And when  $\omega_0^2 \gg \eta^2$

$$\omega = \sqrt{\omega_0^2 - \eta^2} = \omega_0 \left(1 - \frac{\eta^2}{\omega_0^2}\right)^{\frac{1}{2}} \approx \omega_0$$

Then

$$I_{ct} \approx \frac{V_0}{\omega_0 L} e^{-\eta t} \sin \omega_0 t = V_0 \sqrt{\frac{C}{L}} e^{-\frac{\eta}{\omega_0} t} \sin \frac{1}{\omega_0} t$$

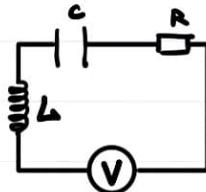
We can sketch it:



## ⑤. LCR circuit with external power source.

According to the Kirchhoff's laws

$$\frac{dV_{ct}}{dt} + L \frac{dI_{ct}}{dt} + RI_{ct} = V_{ct}$$



Then.

$$L \frac{d^2 I_{ct}}{dt^2} + R \frac{dI_{ct}}{dt} + \frac{1}{C} I_{ct} = f_{ct}$$

$$\textcircled{*} \quad \left( \frac{d^2 I_{ct}}{dt^2} + 2\eta \frac{dI_{ct}}{dt} + \omega_0^2 I_{ct} = f_{ct}, \quad \eta = \frac{R}{2L}, \quad \omega_0^2 = \frac{1}{LC}, \quad f_{ct} = \frac{1}{L} \frac{dV_{ct}}{dt} \right)$$

Before solve this question, we have known that the solution of

$$L \frac{d^2 I_{ct}}{dt^2} + R \frac{dI_{ct}}{dt} + \frac{1}{C} I_{ct} = 0$$

$$\left( \frac{d^2 I_{ct}}{dt^2} + 2\eta \frac{dI_{ct}}{dt} + \omega_0^2 = 0, \quad \eta = \frac{R}{2L}, \quad \omega_0^2 = \frac{1}{LC} \right)$$

is i)  $\eta^2 < \omega_0^2$

$$I_{ct} = e^{-\eta t} (A \cos \omega_0 t + B \sin \omega_0 t)$$

ii)  $\eta^2 = 0$

$$I_{ct} = e^{-\eta t} (A + Bt)$$

iii)  $\eta^2 > \omega_0^2$

$$I_{ct} = e^{-\eta t} (A e^{-\omega_0 t} + B e^{\omega_0 t})$$

Then, if we can find a special solution that satisfies

$$\frac{d^2 I_s(\omega)}{dt^2} + 2\eta \frac{dI_s(\omega)}{dt} + \omega_0^2 I_s(\omega) = f(t)$$

we can write the final solution of (4) in the form of  
 $I(t) = I_1(t) + I_2(t)$ .

So. the problem becomes to : try to find a special solution of  $\frac{d^2 I_{1(2)}}{dt^2} + 2\eta \frac{dI_{1(2)}}{dt} + \omega_0^2 I_{1(2)} = f(t)$

Here. let  $I_1(t), I_2(t)$  are two independent solution of the equation  $\frac{d^2 I_{1(2)}}{dt^2} + 2\eta \frac{dI_{1(2)}}{dt} + \omega_0^2 I_{1(2)} = 0$

And the solution is  $I(t) = A I_1(t) + B I_2(t)$

(For example, when  $\eta^2 < \omega_0^2$   $I_1 = e^{-\eta t} \cos \omega_0 t$   
 $I_2 = e^{-\eta t} \sin \omega_0 t$ )

and the solution is  $I = A I_1 + B I_2$

Then we can guess the solution of (4) can be written in a form :  $I_s(t) = A_1(t) I_1(t) + A_2(t) I_2(t)$

and  $\dot{I}_s(t) = \dot{A}_1(t) I_1(t) + \dot{A}_2(t) I_2(t) + A_1(t) \dot{I}_1(t) + A_2(t) \dot{I}_2(t)$

Now, because we just want to find the special solution, we can give some restrictions on  $A_1(t)$  and  $A_2(t)$  to solve it.

Let  $\dot{A}_1(t) I_1(t) + \dot{A}_2(t) I_2(t) = 0$

Then  $\dot{I}_s(t) = A_1(t) \dot{I}_1(t) + A_2(t) \dot{I}_2(t)$

$\ddot{I}_s(t) = \dot{A}_1(t) \dot{I}_1(t) + \dot{A}_2(t) \dot{I}_2(t) + A_1(t) \ddot{I}_1(t) + A_2(t) \ddot{I}_2(t)$

Because  $I_{s(t)}$  is the solution of  $\textcircled{2}$

Then

$$\begin{aligned}
 f(t) &= \underbrace{\dot{A}_{1(t)} I_{1(t)} + \dot{A}_{2(t)} I_{2(t)} + A_{1(t)} \ddot{I}_{1(t)} + A_{2(t)} \ddot{I}_{2(t)}}_{\stackrel{\text{I}_{s(t)}}{\rightarrow}} \\
 &\quad + 2\eta (\underbrace{A_{1(t)} I_{1(t)} + A_{2(t)} I_{2(t)}}_{\stackrel{\dot{I}_{s(t)}}{\rightarrow}}) \\
 &\quad + \omega_0^2 (\underbrace{A_{1(t)} I_{1(t)} + A_{2(t)} I_{2(t)}}_{\stackrel{I_{s(t)}}{\rightarrow}})
 \end{aligned}$$

$$\begin{aligned}
 &= \dot{A}_{1(t)} I_{1(t)} + \dot{A}_{2(t)} I_{2(t)} \quad ? \\
 &\quad + A_{1(t)} \left[ \dot{I}_{1(t)} + 2\eta I_{1(t)} + \omega_0^2 I_{1(t)} \right] \\
 &\quad + A_{2(t)} \left[ \dot{I}_{2(t)} + 2\eta I_{2(t)} + \omega_0^2 I_{2(t)} \right]
 \end{aligned}$$

Then, we get the equation of  $A_{1(t)}$  and  $A_{2(t)}$

$$\begin{cases} \dot{A}_{1(t)} I_{1(t)} + \dot{A}_{2(t)} I_{2(t)} = f(t) \\ \dot{A}_{1(t)} I_{1(t)} + \dot{A}_{2(t)} I_{2(t)} = 0 \end{cases}$$

And we can get

$$\begin{cases} A_{1(t)} = - \int_0^t dt' \frac{f(t') I_{2(t')}}{W(t')} \\ A_{2(t)} = \int_0^t dt' \frac{f(t') I_{1(t')}}{W(t')} \end{cases}, \quad W(t) = I_{1(t)} I_{2(t)} - I_{1(t)} I_{2(t)}$$

Thus, the special solution is

$$I_{s(t)} = - I_{1(t)} \int_0^t dt' \frac{f(t') I_{2(t')}}{W(t')} + I_{2(t)} \int_0^t dt' \frac{f(t') I_{1(t')}}{W(t')}$$

It's easy to confirm it.

$$\begin{aligned}
 \dot{I}_{s(t)} &= - \dot{I}_{1(t)} \int_0^t dt' \frac{f(t') I_{2(t')}}{W(t')} - \frac{f(t) I_{1(t)} I_{2(t)}}{W(t)} + \dot{I}_{2(t)} \int_0^t dt' \frac{f(t') I_{1(t')}}{W(t')} \\
 &\quad + \frac{f(t) I_{1(t)} I_{2(t)}}{W(t)} \\
 &= - \dot{I}_{1(t)} \int_0^t dt' \frac{f(t') I_{2(t')}}{W(t)} + \dot{I}_{2(t)} \int_0^t dt' \frac{f(t') I_{1(t')}}{W(t)}
 \end{aligned}$$

$$\begin{aligned}\ddot{I}_{s(t)} &= -\ddot{I}_{1(t)} \int_0^t \frac{f(t') I_{2(t')}}{W(t')} - \ddot{I}_{1(t)} \frac{f(t) I_{2(t)}}{W(t)} + \ddot{I}_{2(t)} \int_0^t \frac{f(t') I_{1(t')}}{W(t')} + \ddot{I}_{2(t)} \frac{f(t) I_{1(t)}}{W(t)} \\ &= -\ddot{I}_{1(t)} \int_0^t \frac{f(t') I_{2(t')}}{W(t')} + \ddot{I}_{2(t)} \int_0^t \frac{f(t') I_{1(t')}}{W(t')} + \underbrace{\frac{I_1 \dot{I}_2 - I_2 \dot{I}_1}{W(t)}}_{=1} f(t)\end{aligned}$$

$$So \quad \ddot{I}_{s(t)} + 2\eta \dot{I}_{s(t)} + \omega_0^2 I_{s(t)}$$

$$\begin{aligned}&= f(t) - (\ddot{I}_{1(t)} + 2\eta \dot{I}_{1(t)} + \omega_0^2 I_{1(t)}) \int_0^t \frac{f(t') I_{2(t')}}{W(t')} \\ &\quad + (\ddot{I}_{2(t)} + 2\eta \dot{I}_{2(t)} + \omega_0^2 I_{2(t)}) \int_0^t \frac{f(t') I_{1(t')}}{W(t')} \\ &= f(t)\end{aligned}$$

And the general solution is:

$$I_{s(t)} = I_{0(t)} + I_{s(t)}$$

$$= A_1 I_{1(t)} + A_2 I_{2(t)} - I_{1(t)} \int_0^t \frac{f(t') I_{2(t')}}{W(t')} + I_{2(t)} \int_0^t \frac{f(t') I_{1(t')}}{W(t')}$$

To be specific, let  $V_{s(t)} = V_0 \cos \omega t$  and  $\eta^2 < \omega^2$

the equation is

$$L \frac{d^2 I_{s(t)}}{dt^2} + R \frac{d I_{s(t)}}{dt} + \frac{1}{C} I_{s(t)} = \frac{d}{dt} V_0 \cos \omega t = \omega V_0 \sin \omega t$$

$$\Rightarrow \frac{d^2 I_{s(t)}}{dt^2} + 2\eta \frac{d I_{s(t)}}{dt} + \omega^2 I_{s(t)} = f(t) = \frac{\omega V_0}{L} \sin \omega t$$

Then, it is convenient to use the complex space to solve it

$$\Rightarrow \frac{d^2 I_{s(t)}}{dt^2} + 2\eta \frac{d I_{s(t)}}{dt} + \omega^2 I_{s(t)} = f(t) = \frac{V_0}{L} \frac{d}{dt} e^{i\omega t} = \frac{i\omega V_0}{L} e^{i\omega t}$$

Then, according to ④, we can know

$$I_{1(t)} = e^{\lambda_1 t} \quad ; \quad \lambda_1 = -\eta + i\omega' \quad ; \quad \omega' = \sqrt{\omega_0^2 - \eta^2}$$

$$I_{2(t)} = e^{\lambda_2 t} \quad ; \quad \lambda_2 = -\eta - i\omega'$$

$$W(t) = I_1(t) \bar{I}_2(t) - I_1(t) \bar{I}_2(t)$$

$$= \lambda_2 e^{(\lambda_1 + \lambda_2)t} - \lambda_1 e^{(\lambda_1 + \lambda_2)t} = -2i\omega' e^{(\lambda_1 + \lambda_2)t}$$

Then  $\int_0^t \frac{I_1(t') f(t')}{W(t')} dt' = \int_0^t \frac{e^{\lambda_1 t'} \frac{i\omega V_0}{L} e^{i\omega t'}}{-2i\omega' e^{(\lambda_1 + \lambda_2)t'}} dt'$

$$= -\frac{\omega' V_0}{2\omega' L} \int_0^t e^{(i\omega - \lambda_1)t'} dt' \\ = -\frac{\omega V_0}{2\omega L} \frac{1}{i\omega - \lambda_1} (e^{(i\omega - \lambda_1)t} - 1)$$

$$\int_0^t \frac{I_1(t') f(t')}{W(t')} dt' = \int_0^t \frac{e^{\lambda_2 t'} \frac{i\omega V_0}{L} e^{i\omega t'}}{-2\eta e^{(\lambda_1 + \lambda_2)t'}} dt'$$

$$= -\frac{\omega V_0}{2\omega L} \int_0^t e^{(i\omega - \lambda_2)t'} dt'$$

$$= -\frac{\omega V_0}{2\omega L} \frac{1}{i\omega - \lambda_2} (e^{(i\omega - \lambda_2)t} - 1)$$

So the general solution of ④ is

$$I(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \frac{\omega V_0}{2\omega L} \frac{1}{i\omega - \lambda_1} e^{i\omega t} - \frac{\omega V_0}{2\omega L} \frac{1}{i\omega - \lambda_2} e^{i\omega t}$$

$$= e^{-\eta t} (A_1 e^{i\omega t} + A_2 e^{-i\omega t}) + \frac{\omega V_0}{2\omega L} e^{i\omega t} \left[ \frac{1}{i(\omega - \omega^2) + \eta} - \frac{1}{i(\omega + \omega^2) + \eta} \right]$$

$$= e^{-\eta t} (A_1 e^{i\omega t} + A_2 e^{-i\omega t}) + \frac{\omega V_0}{2\omega L} e^{i\omega t} \frac{2i\omega'}{\eta^2 - \omega^2 + \omega'^2 + 2\eta\omega};$$

$$= e^{-\eta t} (A_1 e^{i\omega t} + A_2 e^{-i\omega t}) + \frac{V_0}{L} e^{i\omega t} \frac{i\omega}{-\omega^2 + \omega_0^2 + 2\eta\omega i}$$

$$= e^{-\eta t} (A_1 e^{i\omega t} + A_2 e^{-i\omega t}) + V_0 e^{i\omega t} \frac{1}{\frac{L}{i\omega} (\frac{1}{Lc} - \omega^2 + 2\eta\omega i)}$$

$$= e^{-\eta t} (A_1 e^{i\omega t} + A_2 e^{-i\omega t}) + V_0 e^{i\omega t} \frac{1}{R + i(\omega L - \frac{1}{\omega c})}$$

Thus, when  $t \rightarrow \infty$ , the first part decays fastly

$$\text{And } I(t) \Rightarrow \frac{V_o}{R + i(\omega L - \frac{1}{\omega C})} e^{i\omega t}$$

In fact, this result is obvious, because according to ①~③ we can know the resistance of R, L and C are

$$Z_R = R, \quad Z_L = i\omega L, \quad Z_C = \frac{1}{i\omega C}$$

so in the complex space. the total resistance is

$$Z = Z_R + Z_L + Z_C = R + i(\omega L - \frac{1}{\omega C})$$

$$\text{and } I_{\omega} = \frac{V_o}{Z} = \frac{V_o}{R + i(\omega L - \frac{1}{\omega C})} e^{i\omega t}$$