

Homework 5

Exercise 1 For $n \in \mathbb{N}$ let us set $p_{\frac{1}{n}} : (0, \infty) \rightarrow \mathbb{R}$ for the function defined by $p_{\frac{1}{n}}(x) := x^{\frac{1}{n}}$. If $m \in \mathbb{N}$ we also set $p_{\frac{m}{n}} : (0, \infty) \rightarrow \mathbb{R}$ by $p_{\frac{m}{n}}(x) \equiv x^{\frac{m}{n}} := (x^m)^{\frac{1}{n}} = (x^{\frac{1}{n}})^m$. Finally, for $q \in \mathbb{Q}_+$ we define the function $p_{-q} : (0, \infty) \rightarrow \mathbb{R}$ by $p_{-q}(x) \equiv x^{-q} := \frac{1}{x^q}$.

1) Show that the following equality holds:

$$p'_{\frac{1}{n}}(x) = \frac{1}{n} x^{\frac{1}{n}-1}.$$

For the proof you can use the equality

$$(a^n - b^n) = (a - b) \sum_{k=0}^{n-1} a^{n-k-1} b^k$$

for $a = (x+h)^{\frac{1}{n}}$ and $b = x^{\frac{1}{n}}$. Other arguments which do not involve this formula are also possible.

2) For $m, n \in \mathbb{N}$, deduce that

$$p'_{\frac{m}{n}}(x) = \frac{m}{n} x^{\frac{m}{n}-1}.$$

3) For any $q \in \mathbb{Q}_+$, show that

$$p'_{-q}(x) = -qx^{-q-1}.$$

Conclude that the equality $p'_q = qp_{q-1}$ holds for any $q \in \mathbb{Q}$.

Exercise 2 Compute and simplify the derivative of the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ defined for $x \in \mathbb{R}$ by

$$a) \sin((2x^2 - 3)^2) \quad b) \frac{(x+3)^3}{(2x-3)^2 + 1} \quad c) \frac{1}{\sin^2(3x) + 1}$$

Exercise 3 Compute the derivatives of order 1, 2 and 3 for the functions $f : \mathbb{R} \rightarrow \mathbb{R}$ defined for $x \in \mathbb{R}$ by

$$a) \cos(x) \quad b) \cos(x) \sin(x) \quad c) x^4 + x^3 + x^2 + x^1 + 1$$

Exercise 4 Compute the following limits:

$$(i) \lim_{x \rightarrow 0} \frac{x - \sin(x)}{x^3},$$

$$(ii) \lim_{x \rightarrow 0} \frac{x^2}{1+x-e^x}.$$

Exercise 5 Find the critical points for the differentiable functions $f : \mathbb{R} \rightarrow \mathbb{R}$ defined for $x \in \mathbb{R}$ by

$$a) -x^2 + 2x + 2, \quad b) x^3 - 3, \quad c) \cos(x), \quad d) \sin(x) + \cos(x).$$