

**Homework 13**

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**Exercise 1** *Prove the following statement (ratio test) :*

Let  $(a_j)_{j=1}^{\infty}$  be a sequence with positive terms only. Assume that there exists  $c \in (0, 1)$  and  $N \in \mathbb{N}$  such that

$$\frac{a_{j+1}}{a_j} \leq c \quad \forall j \geq N.$$

Then the corresponding series  $a_1 + a_2 + a_3 + \dots$  is convergent.

**Exercise 2** 1) *Show that the series with generic term  $a_j = \frac{j}{3^j}$  is convergent, in other terms show that*

$$\sum_{j=1}^{\infty} \frac{j}{3^j} < \infty.$$

2) *Show that the series with generic term  $a_j = (-1)^j \frac{j^3}{3^j}$  is absolutely convergent.*

**Exercise 3** *For  $f : [1, \infty) \rightarrow \mathbb{R}_+$  decreasing, prove the following statement (integral test) :*

The series  $f(1) + f(2) + f(3) + \dots$  is convergent if and only if  $\lim_{M \rightarrow \infty} \int_1^M f(x) dx$  is convergent, or in a more simple form show that

$$\sum_{j=1}^{\infty} f(j) < \infty \iff \int_1^{\infty} f(x) dx < \infty.$$

**Exercise 4** *For any  $\varepsilon > 0$  show that the series with generic term  $a_j = j^{-1-\varepsilon}$  is convergent.*